Math 620X Lecture Notes

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Contents

1 January 8th, 2018

Manifolds *

Definition. Let $U, V \subseteq \mathbb{R}^n$ be open sets. Then a map $f\colon U \to V$ is a *diffeomorphism* if f is differentiable and invertible, and f^{-1} is differentiable.

Definition. A subset M of $\mathbb{R}^n$ is a k-dimensional *manifold* (abbreviated mfd) if $\forall x \in M$: $\exists$ open sets $U, V \subseteq \mathbb{R}^n$ with $x \in U$ and $\exists$ a diffeomorphism $h\colon U \to V$ such that

$$h(U \cap M) = V \cap (\mathbb{R}^k \times \{0\}) = \{y \in \mathbb{R}^n \mid y_{k+1} = \cdots = y_n = 0\}$$

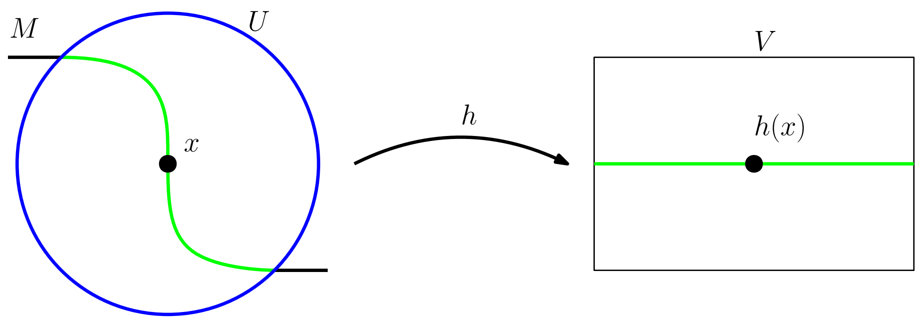


Figure 1: Manifold visualization

Example 1.1. $\mathbb{R}^n$ is an n-dimensional manifold.

Example 1.2. Any open subset of $\mathbb{R}^n$ is an n-dimensional manifold.

Example 1.3. Any singleton $\{x\}$ is a zero dimensional manifold.

Example 1.4. $S^n = \{x \in \mathbb{R}^{n+1} \mid |x|^2 = 1\}$

Theorem 1.5 (Implicit Function Theorem)**.** *Let $A \subseteq \mathbb{R}^n$ be an open subset and $g\colon A \to \mathbb{R}^p$ be a differentiable function such that $g'(x) = \left(\frac{\partial g_i}{\partial x_j}\right)_{ij}$ has rank p whenever $g(x) = 0$. Then $g^{-1}(\{0\})$ is an $(n-p)$-dimensional manifold.*

Example 1.6. Let $g\colon \mathbb{R}^{n+1} \to \mathbb{R}$ defined by $g(x) = |x|^2 - 1$. Then $g'(x) = \nabla g(x) = \begin{bmatrix} 2x_1 & 2x_2 & \cdots & 2x_n \end{bmatrix}$ It is clear that $g'(x) \neq 0$ (thus has rank 1), so by Theorem 1.5 $g^{-1}(\{0\}) = S^n$ is an n-dimensional manifold.

Definition. A *morphism* (or *differentiable map*) $f\colon M \to N$ of manifolds M and N, of dimension k and ℓ respectively, is a function such that $\forall x \in M$ and diffeomorphisms $h\colon U \to V$ and $g\colon U' \to V'$, $\tilde{f} = g \circ f \circ h^{-1}\colon \mathbb{R}^k \times \{0\} \to \mathbb{R}^\ell \times \{0\}$ is differentiable.

*Based on the work in Spivak's *Calculus on Manifolds*, Ch. 5

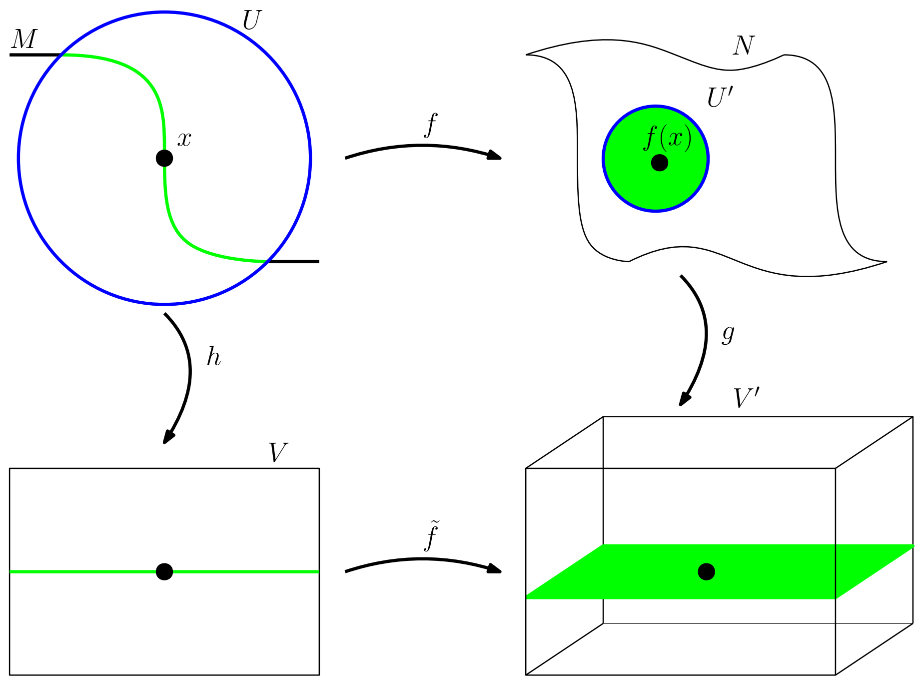


Figure 2: Morphism visualization

Note. If $M \subseteq \mathbb{R}^n$ and $N \subset \mathbb{R}^m$ are manifold, then so is $M \times N$ in a natural way (in $\mathbb{R}^{n+m}$).

Definition. A *Lie group* G is a group which is also a manifold such that the maps,

$$G \times G \to G \text{ by } (g, h) \mapsto gh$$

and

$$G \to G \text{ by } g \mapsto g^{-1}$$

are differentiable maps (i.e morphisms of manifolds).

Example 1.7. $(\mathbb{R}^n, +)$ is a Lie group

Example 1.8. $S^1 = U(1) = \{z \in \mathbb{C} \mid |z| = 1\}$ is a 1-dimensional Lie group with respect to multiplication.

Example 1.9. $SU(2) = \{A \in M_2(\mathbb{C}) \mid AA^* = A^*A = I, \text{ and } \det(A) = 1\}$ is a 3-dimensional Lie group. One can prove that $SU(n)$ is a Lie group for $n \geqslant 2$.

Example 1.10. $GL(n, \mathbb{R}) = \{A \in M_n(\mathbb{R}) \mid \det(A) \neq 0\}$ is an open subset of $M_n(\mathbb{R}) = \mathbb{R}^{n^2}$, so it is a manifold.

Why is it open? $\det\colon \mathbb{R}^{n^2} \to \mathbb{R}$ is continuous which implies that $\det^{-1}(\mathbb{R}\backslash\{0\})$ is open in $\mathbb{R}^{n^2}$.

To construct *Complex Lie Groups*, replace "differentiable" with "complex analytic".

2 January 10th, 2018

Connectedness

Manifolds

Let M be a manifold. Define a binary relation $\sim$ on M by $\forall x, y \in M$:

$$x \sim y \Leftrightarrow \exists \text{ continuous map } \gamma\colon [0,1] \to M \text{ with } \gamma(0) = x, \gamma(1) = y$$

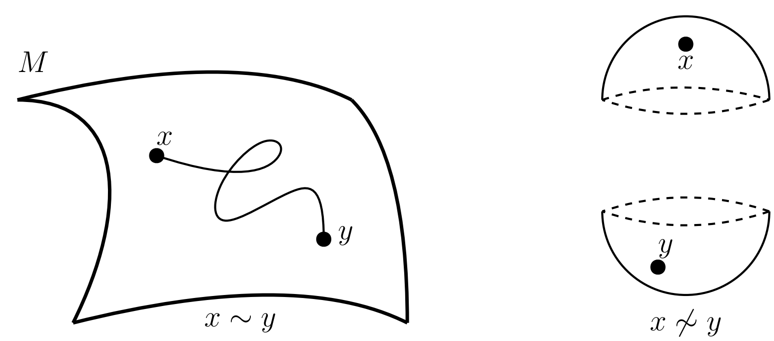


Figure 3: An example and nonexample

If $x \sim y$ we say that x *is connected to* y.

Exercise 2.1. Prove $\sim$ is an equivalence relation.

The equivalence classes

$$[x] = \{y \in M \mid y \sim x\}$$

are *connected components*. The set of equivalence classes of $M/\sim$ is denoted by $\pi_0(M)$. If $x \sim y$ for all $x, y \in M$ we say that M is *connected*.

Example 2.2. S^n is a connected manifold.

Proof. Let $x, y \in S^n$. WLOG $x + y \neq 0$ by connecting them to any antipodal point and composing the maps. Define $\gamma\colon [0,1] \to S^n$ by

$$\gamma(t) = \frac{(1-t)x + ty}{|(1-t)x + ty|}$$

Where the numerator is a line segment in $\mathbb{R}^{n+1}$ and the denominator forces $|\gamma(t)| = 1$. $\square$

Example 2.3. The n-dimensional hyperboloid $H^n \subseteq \mathbb{R}^{n+1}\colon x_1^2 + \cdots + x_n^2 = x_{n+1}^2 - 1$ is an n-dimensional manifold with two connected components.

Exercise 2.4. Prove the statements in the previous example.

Definition. A subset N of a manifold $M \subseteq \mathbb{R}^n$ is an *open submanifold* if there is an open subset U of $\mathbb{R}^n$ such that $N = U \cap M$.

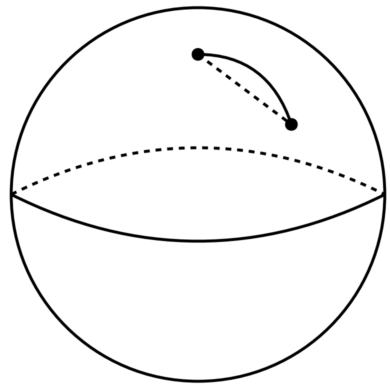


Figure 4: Visualization of γ

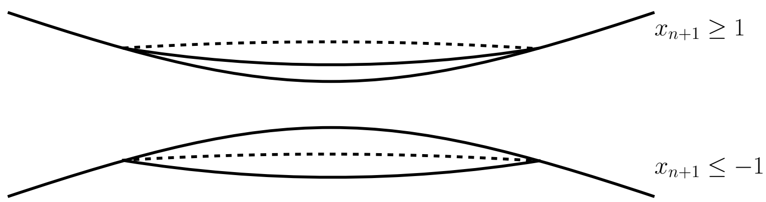


Figure 5: H^n

Proposition 2.5. *Let $M \subseteq \mathbb{R}^n$ be a manifold and $x \in M$. Then the connected component $[x]$ is an open submanifold.*

Proof. For any $y \in [x]$ pick an open set $U_y \subseteq \mathbb{R}^n$ as in the definition of a manifold. Then any $z \in M \cap U_y$ is connected to y, hence to x (see figure 6). So $M \cap U_y \subseteq [x]$. Let $U = \bigcup_{y \in [x]} U_y$. Then U is open in $\mathbb{R}^n$ and $U \cap M = [x]$. ▫

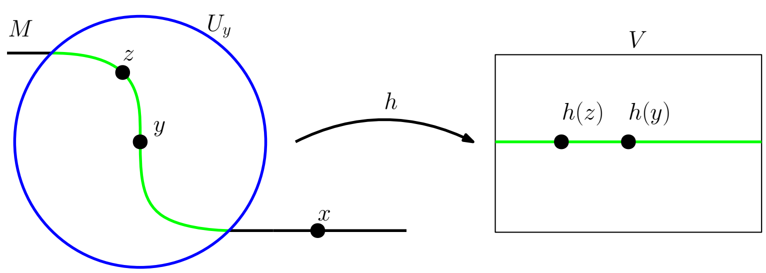


Figure 6: Visual justification that for any $z \in M \cap U_y$, $z \sim y$

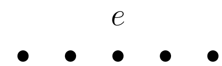
Lie Groups

Definition. A Lie group G is *connected* if it is connected as a manifold.

Notation. If G is a Lie group then G^0 denotes the connected component of G that contains the identity element. $G^0 = [e_G]$.

Remark 1. By proposition 2.5 above, G^0 is an open submanifold of G.

Remark 1. Any finite group G is a 0-dimensional Lie group



$G^0 = \{e\}$.

Theorem 2.6. *Let G be a real or complex Lie group. Then G^0 is a normal subgroup and is itself a Lie group. The quotient group G/G^0 is discrete (i.e. each coset gG^0 is an open submanifold of G).*

Proof. $e \in G^0$ by definition. If $f\colon M \to N$ is continuous then $x \sim y$ in $M \Rightarrow f(x) \sim f(y)$ in N (Exercise). In particular, $f([x]) \subseteq [f(x)]$. Apply to $i\colon G \to G$, $i(g) = g^{-1}$ gives $i(G^0) \subseteq G^0$. Similarly, $m\colon G \times G \to G$, $m(g,h) = gh \Rightarrow m(G^0 \times G^0) \subseteq [m(e,e)] = G^0$. Lastly, fix $g \in G$, $c(h) = ghg^{-1}$ Then $c(G^0)) \subseteq [c(e)] = [e] = G^0$. Thus G^0 is a normal subgroup. Since $gG^0 = [g]$m each coset in G/G^0 is an open submanifold by proposition 2.5, so G/G^0 is discrete. ▫

Example 2.7. $O(n.\mathbb{R}) = \{A \in M_n(\mathbb{R}) \mid A^T A = AA^T = I\}$ has two connected components, orientation-preserving and orientation-reversing, because $A \in O(n, \mathbb{R}) \Rightarrow \det(A) = \pm 1$. So $SO(n, \mathbb{R}) = O(n, \mathbb{R})^0 = \{A \in O(n, \mathbb{R}) \mid \det(A) = 1\}$.

3 January 12th, 2018

Simple Connectedness

Manifolds

Let M be a connected manifold and fix $x_0 \in M$, called a *base point*.

Definition. A *path* in M is a continuous map $\gamma\colon [0,1] \to M$ such that $\gamma(0) = x_0$. γ is a

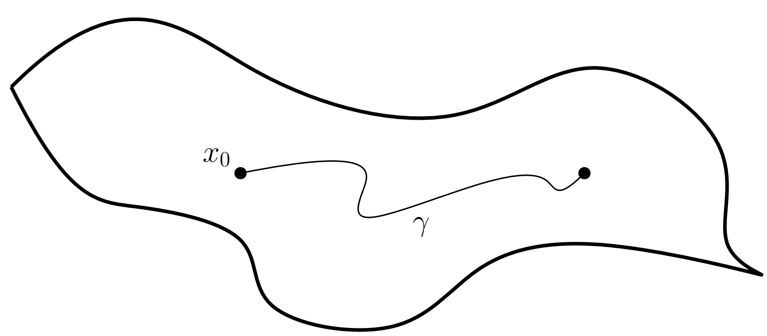


Figure 7: Example of a path γ with base point x_0

loop if $\gamma(0) = \gamma(1)$. The *constant loop* γ_0 is given by $\gamma_0(t) = x_0$ $\forall t \in [0,1]$.

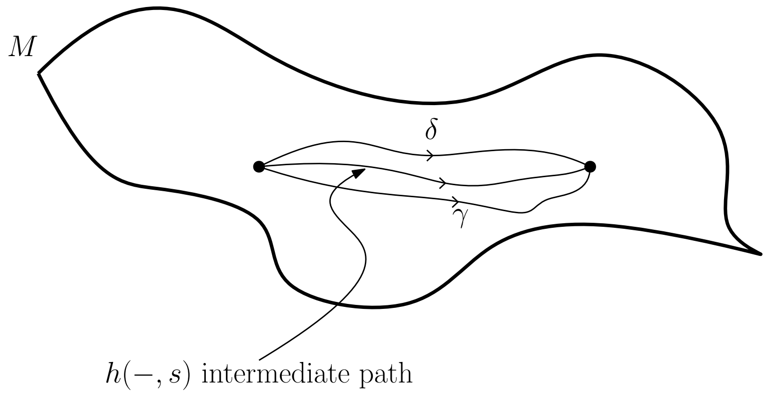


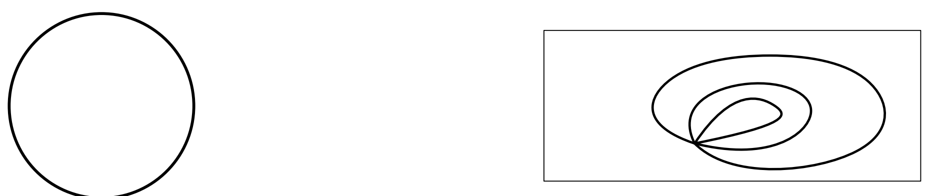
Figure 8: Example of homotopic paths

Definition. Two paths in M γ, δ are *homotopic* if $\gamma(0) = \delta(0), \gamma(1) = \delta(1)$ and $\exists$ continuous map $h\colon [0,1]^2 \to M$ such that

$$h(t,0) = \gamma(t), h(t,1) = \delta(t) \text{ for all } t \in [0,1]$$

Definition. M is *simply connected* if every loop in M is homotopic to the constant loop. (Note: this is independent of the choice of x_0)

Example 3.1. In figure 9 we see that $\mathbb{R}^2$ is a simply connected manifold, while S^1 is not.



(a) S^1 (b) $\mathbb{R}^2$

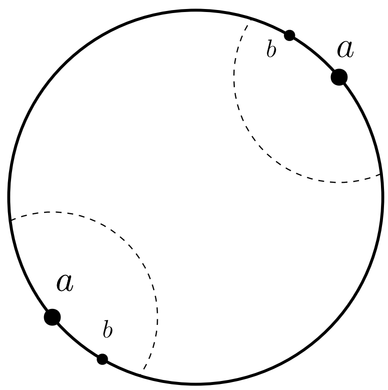
Figure 9: An example and non-example of simply connected manifolds

Example 3.2. The *projective plane* $\mathbb{RP}^2$ (in the Poincaré model) is D^1 but opposite points on S^1 identified: $D^1/\sim$ where $x \sim y$ iff $|x| = |y| = 1$ & $x + y = 0$. Then $\mathbb{P}^1$ is <u>not</u> simply connected. See figures 10a and 10b.

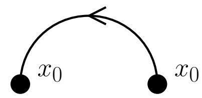
Definition. The *product* of two loops γ, δ in M is $\gamma * \delta\colon [0,1] \to M$

$$\gamma * \delta(t) = \begin{cases} \gamma(2t) & 0 \leqslant t \leqslant \frac{1}{2} \\ \delta(2(t - \frac{1}{2})) & \frac{1}{2} \leqslant t \leqslant 1 \end{cases}$$

Exercise 3.3. Homotopy defines on equivalence relation on the set of loops in M. The set of equivalence classes is denoted $\pi_1(M, x_0)$.



(a) Depiction of $\mathbb{RP}^2$ with identical points labeled



(b) A loop not homotopic to γ_0

Figure 10

Exercise 3.4. $\pi_1(M, x_0)$ is a group with respect to the operation:

$$[\gamma][\delta] = [\gamma * \delta].$$

$\pi_1(M, x_0)$ is the (1st) *fundamental group* of M (a.k.a. *Poincaré group* of M).

Exercise 3.5. $\pi_1(M, x_0) \cong \pi_1(M, y_0)$ for any $x_0, y_0 \in M$ (Recall: we assume M connected).

Example 3.6. $\pi_1(\mathbb{P}^2) \cong \mathbb{Z}/2\mathbb{Z}$, notice $[\gamma]^2 = [\gamma_0]$ in figure 11

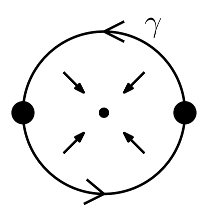


Figure 11

Example 3.7. $\pi_1(S^1) \cong \mathbb{Z}$. The correspondence is the winding number.

Covering Space

Definition. Let M be a connected manifold. A *cover* $(\tilde{M}, p)$ (covering space) for M is a connected manifold M together with a morphism $p\colon \tilde{M} \to M$ such that: $\forall x \in M$, $\exists$ connected open set $U \ni x$ such that every connected component of $p^{-1}(U \cap M)$ diffeomorphically onto $U \cap M$. $\tilde{M}$ is a *universal cover* if it is simply connected.

Example 3.8. $(\mathbb{R}, p)$ for S^1 where $p\colon x \mapsto e^{2\pi i x}$.

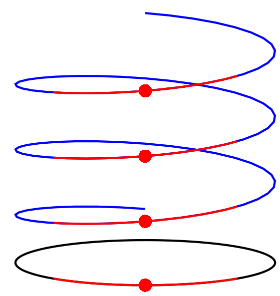


Figure 12: Visualization of $(\mathbb{R}, p)$ as covering space over S^1

Universal Cover $\tilde{M}$

Theorem 3.9. *Every connected manifold has a universal cover. Moreover, it is unique up to diffeomorphism.*

Proof (sketch). Pick a base point $x_0 \in M$. Define $\tilde{M}$ to be the set of homotopy classes of paths in M starting at x_0 (see figure 13) Define $p : \tilde{M} \to M$, $p(\gamma) = \gamma(1)$. One can show that this is a universal cover. ◻

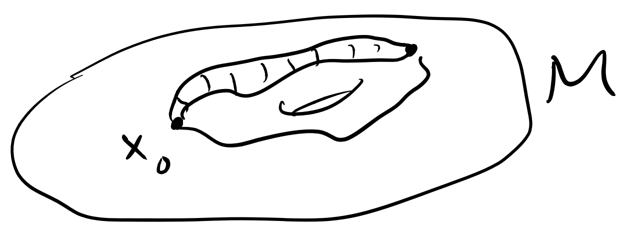


Figure 13: Visualization of Homotopy Classes

Theorem 3.10. *Any morphism of connected manifolds $f\colon M \to N$ can be lifted to $\tilde{f}\colon \tilde{M} \to \tilde{N}$.*

Lie Groups

Definition. A map $\phi\colon G \to H$ of Lie groups is a *morphism of Lie groups* if it is a morphism of manifolds and a group homomorphism.

Theorem 3.11. *If G is a connected (real or complex) Lie group, then its universal cover $\tilde{G}$ has a canonical structure of a Lie group such that*

i) *$p\colon \tilde{G} \to G$ is a morphism of Lie groups*

ii) $\ker p = \pi_1(G, \tilde{e})$ *Moreover, in this case* $\ker p$ *is a discrete subgroup of* $\tilde{G}$*, and* $\ker p \subseteq Z(\tilde{G})$ *the center of* $\tilde{G}$*.*

Example 3.12. $G = S^1 \times S^1 \times \mathbb{Z}$

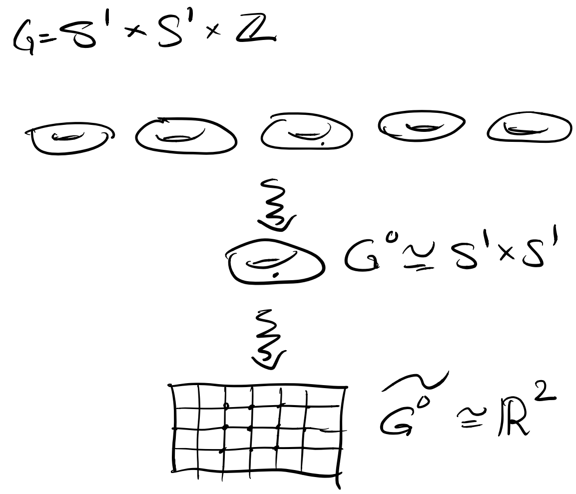


Figure 14: $G = S^1 \times S^1 \times \mathbb{Z}$

4 January 17th, 2018

Theorem 4.1 (Spivak Theorem 5-2 (Coordinate definition of a Manifold))**.** *A subset M of $\mathbb{R}^n$ is a manifold iff $\forall x \in M$ there is an open set $U \subseteq \mathbb{R}^n$, $x \in U$, and an open set $W \subseteq \mathbb{R}^k$ and an injective differentiable function $f : W \to \mathbb{R}^n$ such that*

1) $f(W) = M \cap U$

2) $f'(y)$ *has rank* k $\forall y \in W$

3) $f^{-1} : f(W) \to W$ *is continuous.*

Proof. Read yourself. □

Definition. f is called a *coordinate system around* x.

Tangent Space

Definition. Let M k-dimensional manifold, $x \in M$, $f : W \to \mathbb{R}^n$ be a coordinate system around x, and $a = f^{-1}(x)$. Since $f'(a)$ has rank k, the image of $f'(a) : \mathbb{R}^k \to \mathbb{R}^n$ given by the matrix

$$\begin{bmatrix} \partial_1 f_1(a) & \partial_2 f_1(a) & \cdots & \partial_k f_1(a) \\ \vdots & \vdots & & \vdots \\ \partial_1 f_n(a) & \partial_2 f_n(a) & \cdots & \partial_k f_n(a) \end{bmatrix}$$

is a k-linear subspace of $\mathbb{R}^n$ called the *tangent space of M at the point x*, denoted T_xM.

Often we draw/think of T_xM as the affine space $x + f'(a)\big(\mathbb{R}^n\big)$.

Note. By the chain rule, T_xM is independent of the choice of coordinate system f.

Example 4.2. Describe the tangent space of S^2 at $(0,0,1)$.

Solution. Let $W = \{(s,t) \mid s^2 + t^2 < 1\}$ and $f\colon W \to \mathbb{R}^3$, $f(s,t) = (s,t,\sqrt{1-s^2-t^2})$. Then f is a coordinate system around $x = (0,0,1)$. Let $a = (0,0)$ (notice $f(a) = x$).

$$f'(a) = \begin{bmatrix} \partial_s f_1 & \partial_t f_1 \\ \partial_s f_2 & \partial_t f_2 \\ \partial_s f_3 & \partial_t f_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

so $T_xS^2 = f'(a)\big(\mathbb{R}^2\big) = \mathbb{R}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \oplus \mathbb{R}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

Vector fields

Definition. A *vector field on M* is a map $F\colon M \to \mathbb{R}^n$ such that $F(x) \in T_xM$ for all $x \in M$. F is called *differentiable* if for every coordinate system $f\colon W \to \mathbb{R}^n$ at any $x \in M$, $a \mapsto f'(a)\big(F(f(a))\big)^\dagger$ is a differentiable vector field on W.

Derivative (differential)

If $f\colon M \to N$ is a morphism of manifolds there is an induced linear map $df = f_* = f'(p)\colon T_pM \to T_{f(p)}N$ given as follows

$$\begin{aligned} \alpha\colon U \to \mathbb{R}^n \quad & \alpha(a) = p \\ \beta\colon V \to \mathbb{R}^n \quad & \beta(b) = f(p) \\ T_pM &= \alpha'(a)\big(\mathbb{R}^k\big) \\ T_{f(p)}N &= \beta'(b)\big(\mathbb{R}^\ell\big) \end{aligned}$$

Define $f'(p)$ bey the formula

$$(\beta^{-1} \circ f \circ \alpha)'(a) = (\beta'(b))^{-1} \circ f'(p) \circ \alpha'(a).$$

† this mapping is single-valued

Lie Groups

Fix $g \in G$. There are three maps $G \to G$:

$$\begin{aligned} \mathrm{L}_g &\colon h \mapsto gh \\ \mathrm{R}_g &\colon h \mapsto hg^{-1} \\ \mathrm{Ad}_g &\colon H \mapsto ghg^{-1} \end{aligned}$$

They are morphisms of manifolds $\Rightarrow$ have differentials.

Example 4.3. $d\mathrm{L}_g \colon T_1 G \to T_g G$, $x \mapsto g.x$

Definition. A vector field F on a Lie group G is *left-invariant* if $g.F(x) = F(gx)$ for all $g, x \in G$.

Definition. $\mathfrak{g} = \{$all left-invariant v.fields on $G\}$

Theorem 4.4. *$v \mapsto v(1)$ is linear isomorphism of $\mathfrak{g}$ with $T_1(G)$.*

Proof. Let $x \in T_1(G)$ Define $v(g) = g.x$. Then v is left invariant uniqueness of □

5 January 19th, 2018

Classical groups

$\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

- Genreal Linear group $GL(n, \mathbb{K})$ is the group of invertible $n \times n$ matrices $A = (a_{ij})_{i,j=1}^n$, $a_{ij} \in \mathbb{K}$.
- Special Linear group $SL(n, \mathbb{K}) = \{A \in GL(n, \mathbb{K}) \mid \det A = 1\}$.
- Orthogonal group Let $\langle \cdot, \cdot \rangle$ be the standard non-degenerate symmetric bilinear form on $\mathbb{K}^n$: $\langle \mathbb{e}_i, \mathbb{e}_j \rangle = \delta_{ij}$. $O(n, \mathbb{K}) = \{A \in GL(n, \mathbb{K}) \mid \langle Ax, Ay \rangle = \langle x, y \rangle \ \forall x, y \in \mathbb{K}^n\} = \{A \in GL(n, \mathbb{K}) \mid A^T A = AA^T = I_n\}$.
- Special Orthogonal group $SO(n, \mathbb{K}) = SL(n, \mathbb{K}) \cap O(n, \mathbb{K})$. $SO(p, q; \mathbb{R})$, $p + q = n$. Let $\langle \cdot, \cdot \rangle_{p,q}$ be the bilinear form on $\mathbb{R}^n$ with signature (p, q)

$$\langle x, y \rangle_{p,q} = \sum_{i=1}^{n} x_i y_i - \sum_{p+1}^{p+q} x_i y_i$$

  Then $SO(p, q; \mathbb{R}) = \{A \in GL(n, \mathbb{R}) \mid \langle Ax, Ay \rangle_{p,q} = \langle x, y \rangle_{p,q}\}$.
- Symplectic group On $\mathbb{K}^{2n}$ define bilinear form $\omega(x, y) = \sum_{1 \leqslant i \leqslant n} (x_i y_{i+n} - y_i x_{i+n})$ which is the unique non-degenerate skew-symmetric bilinear form on $\mathbb{K}^{2n}$. $Sp(n, \mathbb{K}) = \{A \in GL(2n, \mathbb{K}) \mid \omega(Ax, Ay) = \omega(x, y)\}$.

Exercise 5.1. Special Symplectic? Show $Sp(n, \mathbb{K}) \subseteq SL(2n, \mathbb{K})$, so it is already "special".

The following important real Lie groups are also considered classical:

- Unitary group $U(n) = \{A \in GL(n, \mathbb{C}) \mid A^*A = AA^* = I_n\}$ Where $A^* = \overline{A^T}$ hermitian adjoint.
- Special unitary group $SU(n) = U(n) \cap SL(n, \mathbb{C})$.
- Unitary quaternionic group $Sp(n) = Sp(n, \mathbb{C}) \cap SU(2n)$.

How to prove these are Lie groups?

Method 1: Implicit function theorem (thm 1.5) $O(n, \mathbb{K}) = \{A = (a_{ij}) \mid AA^T = I_n\}$ is defined by n^2 equations in $\mathbb{K}^{n^2}$. Compute Jacobian of the system and show full rank.

Method 2: Observe that $O(n, \mathbb{K})$ forms a closed subset of $GL(n, \mathbb{K})$ and use Theorem about closed Lie subgroups (next time)

Method 3: Lie algebra and exponential map. $\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges absolutely in matrix norm for any $n \times n$ matrix x. So $\exp\colon \mathfrak{gl}(n, \mathbb{K}) \to \mathfrak{gl}(n, \mathbb{K})$ where $\mathfrak{gl}(n, \mathbb{K}) =$ set of all $n \times n$ matrices. $\log(1 + x) = \sum_{k=1}^{\infty} \frac{(= 1)^{k+1} x^k}{k}$ is analytic near $1 \in \mathfrak{gl}(n, \mathbb{K})$.

Theorem 5.2.

1) $\log(\exp(x)) = x$ *and* $\exp(\log(x)) = x$ *whenever defined.*

2) $\exp(0) = 1,\ d\exp(0) = 1$.

3) *If* $xy = yx$ *then* $\exp(x + y) = \exp(x)\exp(y)$. *If* $xy = yx$ *then* $\log(xy) = \log(x) + \log(y)$ *near 1.*

4) *For fixed* $x \in \mathfrak{gl}(n, \mathbb{K})$ $t \mapsto \exp(tx)$ *is a morphism of Lie groups* $\mathbb{K} \to GL(n, \mathbb{K})$ *(a one-parameter subgroup).*

5) $\exp(AxA^{-1}) = A\exp(x)A^{-1}$ *and* $\exp(x^t) = (\exp(x))^t$

Note. $T_1 GL(n, \mathbb{K}) \cong \mathfrak{gl}(n, \mathbb{K})$.

Theorem 5.3 (Thm 2.30 in Kirillov)**.** *For each classical group* $G \subset GL(n, \mathbb{K})$ *there is a vector space* $\mathfrak{g} \subset \mathfrak{gl}(n, \mathbb{K})$ *such that for some neighborhood* U *of 1 in* $GL(n, \mathbb{K})$ *and some neighborhood* u *of 0 in* $\mathfrak{gl}(n, \mathbb{K})$ *such that*

$$\begin{tikzcd} U \cap G \arrow[rr, bend left, "\log"] & & u \cap \mathfrak{g} \arrow[ll, bend left, "\exp"] \end{tikzcd}$$

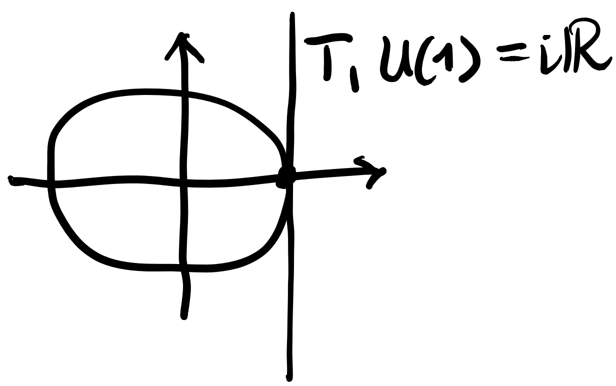
6 January 22nd, 2018

Classical Groups (con't)

First we begin with two examples with tangent spaces.

Example 6.1. $T_a\mathbb{R}^k = \mathbb{R}^k$ because $\mathrm{Id}\colon \mathbb{R}^k \to \mathbb{R}^k$ is a coordinate system around a. $\mathrm{Id}(x_1, \dots, x_k) = (x_1, \dots, x_k)$ and $\mathrm{Id}'(a) = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix} = I_k$ and $I_k(\mathbb{R}^k) = \mathbb{R}^k$. Similarly $T_a\mathbb{C}^k = \mathbb{C}^k$.

Example 6.2. $T_1U(1) = i\mathbb{R} \cong \mathbb{R}$. $U(1) = \{z \in \mathbb{C} \mid zz^\star = 1 = \{z \in \mathbb{C} \mid |z| = 1\}$. Now $\varphi\colon (-\varepsilon, \varepsilon) \to U(1)$ with $\varphi(t) = e^{2\pi i t}$ is a coordinate system around $1 \in U(1)$, $\varphi(0) = 1$, $\varphi'(0) = 2\pi i$ (1×1 matrix). $\varphi'(0)\big(\mathbb{R}\big) = 2\pi i \cdot \mathbb{R} = i\mathbb{R}$



Definition. A *neighborhood* (nbg) of a point $x \in \mathbb{R}^n$ is an open subset $U \subseteq \mathbb{R}^n$ such that $x \in U$.

Recall from last time, Theorem 5.3. Before we prove this theorem we prove the following corollary:

Corollary 6.3 (Corollary 2.31 in Kirillov)**.** *Each classical group is a Lie group with tangent space $T_1G \cong \mathfrak{g}$ and $\dim G = \dim \mathfrak{g}$.*

Proof of Cor. $\exp\colon u \cap \mathfrak{g} \to U \cap G$ is a coordinate system around $1 \in G$, by Theorem 5.3, and $\exp'(0) = \mathrm{Id}$ which has full rank. Let $g \in G$ be arbitrary then the composition the $\mathrm{L}_g \circ \exp\colon u \cap \mathfrak{g} \to U \cap G$ is a coordinate system around g. So every $g \in G$ has a coordinate system so by theorem 4.1, G is a manifold.

For the second part,

$$\begin{array}{ccccc} \exp_*: & T_0\,\mathfrak{g} & \to & T_1 G \\ \shortparallel & \shortparallel & & \\ d\exp(0) & \mathfrak{g} & & \\ \shortparallel & & & \\ \mathrm{Id} & & & \end{array}$$

□

Proof of thm 5.3.

$\underline{G = GL(n, \mathbb{K})}$

By properties of exp and log.

$\underline{G = SL(n, \mathbb{K})}$

For any $x \in \mathfrak{gl}(n, \mathbb{K})$ we have the identity

$$\exp(\mathrm{Tr}(x)) = \det(\exp(x)) \tag{6.1}$$

For any $A \in GL(n, \mathbb{K})$,

$$\exp(AxA^{-1}) = A\exp(x)A^{-1}$$

Proof of Equation 6.1. So 6.1 holds for x iff it holds for AxA^{-1}. Find $A \in GL(n, \mathbb{C})$ such that $AxA^{-1} = \mathfrak{s} + \mathfrak{n}$ where $\mathfrak{s}, \mathfrak{n} \in \mathfrak{gl}(n, \mathbb{C})$, $\mathfrak{s}$ diagonal, $\mathfrak{n}$ strictly upper triangular, $\mathfrak{s}\mathfrak{n} = \mathfrak{n}\mathfrak{s}$. Then use that

$$\exp(\mathfrak{s} + \mathfrak{n}) = \exp(\mathfrak{s})\exp(\mathfrak{n}).$$

Easy to check 6.1 for $\mathfrak{s}$ and $\mathfrak{n}$ $\Rightarrow$ holds for $\mathfrak{s} + \mathfrak{n}$ $\Rightarrow$ holds for x. □

Thus for $X \in GL(n, \mathbb{K})$ near 1, $X = \exp x$ for some $x \in \mathfrak{gl}(n, \mathbb{K})$.

$$\begin{aligned} \det(X) &= \det(\exp(x)) \\ &= \exp(\mathrm{Tr}(x)). \end{aligned}$$

So $\det(X) = 1 \Leftrightarrow \mathrm{Tr}(x) = 0$, so statement is true with $\mathfrak{g} = \{x \in \mathfrak{gl}(n, \mathbb{K}) \mid \mathrm{Tr}(X) = 0\}$.

Notation. $\mathfrak{sl}(n, \mathbb{K})$.

$\underline{G = O(n, \mathbb{K})} = \{X \in GL(n, \mathbb{K}) \mid XX^t = I\}$

For $X \in GL(n, \mathbb{K})$ near 1 write $X = \exp(x), x \in \mathfrak{gl}(n, \mathbb{K})$ by properties of exp. Then

$$\begin{aligned} XX^t = I \Rightarrow I &= X^{-1}(x^t)^{-1} = (X^tX)^{-1} \\ &\Rightarrow X^tX = XX^t = I \\ &\Rightarrow 0 = \log I = \log X + \log X^t = x + x^t \end{aligned}$$

Conversely if $x + x^t = 0$ then x and x^t commute $\Rightarrow$ $\exp(X)\exp(x^t) = \exp(x + x^t) = \exp(0) = I$. So statement true with $\mathfrak{g} = \mathfrak{o}(n, \mathbb{K}) = \{x \in \mathfrak{gl}(n, \mathbb{K}) \mid x + x^t = 0\}$ set of skew-symmetric matrices.

$\underline{G = SO(n, \mathbb{K})}$

$\mathfrak{g} = \mathfrak{so}(n, \mathbb{K}) = \mathfrak{sl}(n, \mathbb{K}) \cap \mathfrak{o}(n, \mathbb{K})$. However!

$$x + x^t = 0 \Rightarrow \mathrm{Tr}\, x = 0$$

So actually $\mathfrak{so}(n,\mathbb{K}) = \mathfrak{o}(n,\mathbb{K})$. Which makes sense because $SO(n,\mathbb{K}) = O(n,\mathbb{K})^0$ the connected component of I.
$\underline{G = U(n), SU(n)}$
$\mathfrak{u}(n) \neq \mathfrak{su}(n)$
$\underline{G = Sp(n,\mathbb{K})}$
$\mathfrak{sp}(n,\mathbb{K})$
$\underline{G = Sp(n)}$
$\mathfrak{sp}(n)$
Read about the previous few in the book. □

7 January 24th, 2018

Submanifolds

- Open
- Immersed
- Embedded

Recall:
A subset N of a manifold $M \subseteq \mathbb{R}^n$ is an *open submanifold* if $N = M \cap U$ for some open subset U of $\mathbb{R}^n$.

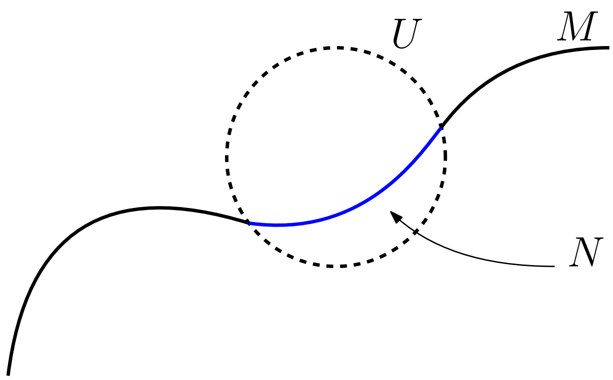


Figure 15: Visualization of an open submanifold

Example 7.1. $M = S^1 \subseteq \mathbb{R}^2$, $U = \{(x,y) \mid y > 0\}$, $N = M \cap U$ is an open submanifold of S^1.

Example 7.2. $GL(n,\mathbb{K})$ is an open submanifold of $\mathbb{K}^{n^2}$.

Example 7.3. Every connected component of a manifold is an open submanifold. In particular G^0 is an open submanifold of G.

Definition. A morphism of manifolds $f\colon X \to Y$ is an *immersion* if $f_*\colon T_xX \to T_{f(x)}Y$ has full rank ($= \dim X$) for every $x \in X$.

Example 7.4. $f\colon S^1 \to \mathbb{R}^2$, $f(\cos\theta, \sin\theta) = (\cos\theta, \sin 2\theta)$

$\varphi\colon \theta \mapsto \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} = x$, $\varphi'(a) = \begin{bmatrix} -\sin a \\ \cos a \end{bmatrix}$, $T_x S^1 = \mathbb{R} \cdot \begin{bmatrix} -\sin a \\ \cos a \end{bmatrix}$, $f_*\colon T_x S^1 \to T_{f(x)}\mathbb{R}^2 = \mathbb{R}^2$ is given by $f_*\left(t \cdot \begin{bmatrix} -\sin a \\ \cos a \end{bmatrix}\right) \overset{(*)}{=} t \cdot \begin{bmatrix} -\sin a \\ 2\cos 2a \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ for $t \neq 0$.

Proof of $(*)$**.**

$$(*)\colon \quad \begin{array}{ccc} M & \xrightarrow{f} & N = \mathbb{R}^2 \\ \alpha \uparrow & & \uparrow \beta = \mathrm{Id} \\ W & \rightarrow & W' = \mathbb{R}^2 \end{array}$$

$(\beta^{-1} f \alpha)' = (\beta^{-1})' f' \alpha'$, so $\beta'(\beta^{-1} f\alpha)' = f'\alpha'$. Now since $\beta = \mathrm{Id}$, $(f\alpha)' = f'\alpha'$

$$\begin{bmatrix} -\sin a \\ 2\cos 2a \end{bmatrix} = f'\left(\begin{bmatrix} -\sin a \\ \cos a \end{bmatrix}\right)$$

□

Definition. The pair (X, f) is an *immersed submanifold* of Y. By abuse of terminology we sometimes say $f(X)$ is an immersed submaniofld.

Definition. If $f\colon X \to Y$ is an immersion such that

1) f is injective

2) $f\colon X \to f(X)$ is a homeomorphsim,

then f is an *embedding* and we say $f(X)$ is an *(embedded) submanifold* of Y.

Example 7.5. $f\colon \mathbb{R} \to S^1 \subset \mathbb{R}^2$ is an immersion

Example 7.6. $f\colon \mathbb{R} \to \mathbb{R}^2$, $t \mapsto (t, \sin t)$ is an embedding. The graph of $y = \sin x$ is a submanifold of $\mathbb{R}^2$.

Example 7.7. $f\colon \mathbb{R} \to \mathbb{R}^2$

$$f(t) = \begin{cases} (0, t+2) & , -\infty < t \leqslant -1 \\ \gamma(t) & , -1 \leqslant t \leqslant 1 \\ (\frac{3}{t^2}, \sin \pi t) & , 1 \leqslant t < \infty \end{cases}$$

making f differentiable and injective. Then f is not an embedding since $f\colon \mathbb{R} \to f(\mathbb{R})$ is not a homeomorphism.

Example 7.8. $f\colon \mathbb{R} \to S^1 \times S^1$, $f(t) = (e^{ati}, e^{bti})$ where $a, b \in \mathbb{R}\backslash\{0\}$ such that a/b is irrational. Then f is an injective immersion, but $f(\mathbb{R})$ is dense in $S^1 \times S^1$ so $f\colon \mathbb{R} \to f(\mathbb{R})$ is not a homeomorphism $\Rightarrow$ f not an embedding.

Definition. A *closed Lie subgroup* H of a Lie group G is a subgroup which is also a submanifold.

Example 7.9. Any linear subspace U of a vector space V is a closed Lie subgroup.

Example 7.10. G^0 is a closed Lie subgroup of G using the indetity map embedding $G^0 \to G$ by $x \mapsto x$.

Example 7.11. If G_1 and G_2 are Lie groups then $G_1 \times \{1\}$ and $\{1\} \times G_2$ are closed Lie subgroups.

Theorem 7.12 (Thm 2.9 in Kirillov)**.**

1) *Let H be a closed Lie subgroup of Lie group G. Then $H = V \cap G$ for some closed subset $V \subseteq \mathbb{R}^n$. (i.e. H is* closed in *G)*

2) *Conversely, any subgroup H of a Lie group G such that H is closed in G, is a closed Lie subgroup.*

Proof. Skipped. ▫

Example 7.13. $Sp(n, \mathbb{K})$ is a closed Lie subgroup of $GL(2n, \mathbb{K})$.

Proof. Know $Sp(n, \mathbb{K})$ subgroup. $\forall A \in GL(2n, \mathbb{K})$: $A \in Sp(n, \mathbb{K})$ iff $\omega(Ax, Ay) = \omega(x, y)$ $\forall x, y \in \mathbb{K}^{2n} \Leftrightarrow (Ax)^t \cdot J \cdot Ay = x^t \cdot J \cdot y$ $\forall x, y \in \mathbb{K}^{2n} \Leftrightarrow A^t JA = J$, $J = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$. So let $V = \{A \in M_{2n}(\mathbb{K}) \mid A^t JA = J\}$. Then V is a closed subset of $M_{2n}(\mathbb{K}) = \mathbb{K}^{(2n)^{2n}}$, and $Sp(2n, \mathbb{K}) = V \cap GL(2n, \mathbb{K})$. Similar for other classical Lie groups. ▫

8 January 26th, 2018

Quotient groups and Homogeneous spaces

[Corollary 2.10, read yourselves]

Definition (Rough idea). Let F be a manifold. A morphism $p\colon T \to B$ of manifolds is a *fiber bundle over B with fiber F* if

1) p is surjective,

2) p is *locally trivial:* Each $x \in B$ has a neighborhood U a neighborhood U and a diffeomorphism $\mathcal{T}_U\colon p^{-1}(U) \to U \times F$ called local trivialization, such that commutes *i.e.*

$$\begin{tikzcd} U \times F \arrow[dr, "pr_1"'] & p^{-1}(U) \subset T \arrow[l, "\mathcal{T}_U"'] \arrow[dl, "p"] \\ & U \end{tikzcd}$$

$p|_{p^{-1}(U)} = pr_1 \circ \mathcal{T}_U$.

3) Whenever $U \cap V \neq \varnothing$, trivializations $\mathcal{T}_U, \mathcal{T}_V$ are compatible in a differentiable way.

B=base space, F=fiber, T=total space.

Example 8.1. Let F, B any manifolds $p\colon F \times B \to B$, $(x, y) \mapsto y$ is the *trivial fiber bundle* over B with fiber F.

Example 8.2. Tangent bundle on S^2

$$TS^2 = \{(x, v) \mid x \in S^2, v \in T_x S^2\}, \quad p : TS^2 \to S^2, \quad (x, v) \mapsto x$$

is a fiber bundle with fiber $\mathbb{R}^2$.

Example 8.3. M any manifold. *Tangent bundle* on M:

$$TM = \{(x, v) \mid x \in M, v \in T_x M\}, \quad p\colon TM \to M, \quad (x, v) \mapsto x$$

is a fiber bundle with fiber $\mathbb{R}^k$, $k = \dim M$.

Theorem 8.4 (Thm 2.11 in Kirillov)**.**

1) *Let G be a Lie group of dimension n, and $H \subset G$ a closed Lie subgroup of dimension k. Then the set of left cosets $G/H = \{gH \mid g \in G\}$ has a natural structure of a manifold of dimension $n - k$, such that the canonical map $p\colon G \to G/H$, $g \mapsto gH$ is a fiber bundle with fiber H. Also $T_{\bar{1}}(G/H) \cong T_1 G/T_1 H$ ($\bar{1} = p(1) = H$).*

2) *If H normal, closed Lie subgroup of a Lie group G, then G/H has a canonical structure of a Lie group, and $p\colon G \to G/H$ gives an isomorphism $T_{\bar{1}}(G/H) = T_1 G/T_1 H$.*

Proof. Beyond scope of class. □

2.3 Homomorphism Thm. Read yourselves.

2.4, 2.5 Homogeneous Spaces

Let M be a manifold. Let $\mathrm{Diff}(M)$ be the group of diffeomorphisms $\varphi\colon M \to M$.

Definition. An *action* of a Lie group G on M is a group homomorphism $\rho\colon G \to \mathrm{Diff}(M)$ such that the map $G \times M \to M$, $(g, x) \mapsto \rho(g)(x)$ is a morphism of manifolds

Notation. $g.x := \rho(g)(x)$

Example 8.5. $GL(n, \mathbb{R})$ acts on $\mathbb{R}^n$

Example 8.6. $O(n, \mathbb{R})$ acts on S^{n-1}.

Example 8.7. G acts on G via $\mathrm{Ad}, \mathrm{L}, \mathrm{R}$ $g \mapsto \mathrm{Ad}\, g$.

Example 8.8. $GL(n, \mathbb{R})$ acts on the set of *flags* in $\mathbb{R}^n$:

$$\mathcal{F}_n(\mathbb{R}) = \{(V_0 \subset V_1 \subset \cdots \subset V_n) \mid V_d \text{ subspace of } \mathbb{R}^n, \dim V_d = d\}$$

Note. $\mathcal{F}_2(\mathbb{R}) \cong \mathbb{RP}^2$ and $g.(V_0 \subset V_1 \subset \cdots \subset V_n) := (gV_0 \subset gV_1 \subset \cdots \subset gV_n)\ \forall g \in GL(n, \mathbb{R})$.

Theorem 8.9 (Thm 2.20 in Kirillov)**.** *Let M be a manifold with an action of a Lie group G. Then $\forall m \in M$, the stabilizer* $\mathrm{Stab}_G(m) = G_m = \{g \in G \mid g.m = m\}$ *is a closed Lie subgroup of G and the map*

$$G/G_m \to M \quad gG_m \mapsto g.m$$

is an injective immersion.

Proof. Future (?) $\square$

Corollary 8.10 (Cor 2.21 in Kirillov)**.** *Each orbit $\mathcal{O}_m := \{g.m \mid g \in G\}$ is an immersed submanifold of M, and $T_m\mathcal{O}_m = T_1G/T_1G_m$. If $\mathcal{O}_m$ is a submanifold then $G/G_m \xrightarrow{\sim} \mathcal{O}_m$ is a diffeomorphism.*

Definition. A *G-homogeneous space* is a manifold with a transitive action of G.

Corollary 8.11. *Let M be a G-homogeneous space and fix $x \in M$. Then $G \to M$, $g \mapsto g.m$ is a fiber bundle with fiber G_m, and $M \cong G/G_m$ as G-homogeneous spaces.*

Example 8.12. $SO(n-1, \mathbb{R}) \to SO(n, \mathbb{R}) \to S^{n-1}$, where $SO(n-1, \mathbb{R})$ is the stabilizer of $m = (0, 0, \ldots, 1)$. So

$$S^{n-1} \cong \frac{SO(n, \mathbb{R})}{SO(n-1, \mathbb{R})}$$

Example 8.13. $GL(n, \mathbb{R})$ acts transitively on $\mathcal{F}_n(\mathbb{R})$. Pick standard flag

$$V^{\text{st}} = (\{0\} \subset \langle \mathbb{e}_1 \rangle \subset \cdots \subset \langle \mathbb{e}_1, \ldots, {}_{n-1} \rangle \subset \mathbb{R}^n).$$

Then the stabilizer $\mathrm{Stab}_{GL(n,\mathbb{R})}(V^{\text{st}}) = B(n, \mathbb{R})$ all invertible upper triangular matrices so

$$\mathcal{F}_n(\mathbb{R}) \cong \frac{GL(n, \mathbb{R})}{B(n, \mathbb{R})}$$

which equips $\mathcal{F}_n(\mathbb{R})$ with the structure of a manifold of dimension $n^2 - \frac{n(n+1)}{2} = \frac{n(n-1)}{2}$

9 January 29th, 2018

The general exponential map

Let G be a Lie group (not necessarily classical) and $\mathfrak{g} = T_1G$ its Lie algebra.

Goal: Define a map $\exp\colon \mathfrak{g} \to G$ which generalizing matrix exponential map.

$$\mathfrak{gl}(n, \mathbb{K}) \to GL(n, \mathbb{K}) \quad x \mapsto e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

The problem is x^k has no meaning in general.

Proposition 9.1 (Prop 3.1 in Kirillov). *Let G be a Lie group, $\mathfrak{g} = T_1G$, and $x \in \mathfrak{g}$. Then there exists a unique morphism of Lie groups*

$$\gamma_x \colon \mathbb{K} \to G \quad t \mapsto \gamma_x(t)$$

such that $\dot{\gamma}_x(0) = x$ where $\dot{\gamma}_x = \dfrac{\partial \gamma_x}{\partial t}$. So $\dot{\gamma}_x(0) \colon T_0\mathbb{K} \to T_1G = \mathfrak{g}$ (Recall: $T_0\mathbb{K} = \mathbb{K}$).

Definition. γ_x is the *one-parameter subgroup* of G corresponding to x.

Definition. The *exponential map* $\exp\colon \mathfrak{g} \to G$ is defined by $\exp(x) = \gamma_x(1)$.

Proof of prop 9.1.

$\mathbb{K} = \mathbb{R}$

Suppose $\gamma\colon \mathbb{R} \to G$ is a morphism of Lie groups such that $\dot{\gamma}(0) = x$. Then

$$\begin{aligned}
\dot{\gamma}(t) &= \lim_{h\to 0} \frac{\gamma(t+h) - \gamma(t)}{h} \\
&= \lim_{h\to 0} \frac{\gamma(t)\gamma(h) - \gamma(t)\gamma(0)}{h} \\
&= \lim_{h\to 0} \frac{(\mathrm{L}_{\gamma(t)} \circ \gamma)(h) - (\mathrm{L}_{\gamma(t)} \circ \gamma)(0)}{h} \\
&= (\mathrm{L}_{\gamma(t)} \circ \gamma)'(0) \\
&= (\mathrm{L}_{\gamma(t)})_*(\dot{\gamma}(0)) \\
&= (\mathrm{L}_{\gamma(t)})_*(x)
\end{aligned}$$

So, this gives a differential equation for γ i.e. $\begin{cases} \dot{\gamma}(t) = (\mathrm{L}_{\gamma(t)})_*(x) \\ \dot{\gamma}(0) = x \end{cases}$. By the theory of ODE's, this equation has a unique solution $\gamma(t)$. □

Example 9.2. $G = U(1) = \{z \in \mathbb{C}^X \mid |z|^2 = 1\}$ is real 1-dim Lie group. We have seen $\mathfrak{g} = i\mathbb{R}$. Given $x \in i\mathbb{R}$, the corresponding 1-parameter subgroup is

$$\gamma_x(t) = e^{tx}$$

because $\gamma_x(s+t) = e^{(s+t)x} = e^{sx}e^{tx} = \gamma_x(s)\gamma_x(t)$ and $\dot{\gamma}_x(t) = xe^{tx}$ so $\dot{\gamma}_x(0) = x$. So the exponential map in this in this case is $i\mathbb{R} \to U(1)$ by $x \mapsto \gamma_x(1) = e^x$ i.e. the usual one.

Example 9.3. $SO(3,\mathbb{R})$ it Lie algebra $\mathfrak{so}(3,\mathbb{R})$ has a basis $\{J_x, J_y, J_z\}$:

$$J_x = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \quad J_y = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \quad J_z = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

One can show that

$$\gamma_{J_x}(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos t & -\sin t \\ 0 & \sin t & \cos t \end{bmatrix}.$$

Indeed it's a morphism of Lie groups and $\dot{\gamma}_{J_x}(0) = J_x$. So $\exp(tJ_x) = \gamma_{J_x}(t)$.

Application to flows of vector fields

Given a vector field v on a manifold, we can regard v as prescribing the velocity of a particle given initial position and time duration $v\colon M \to TM$, $x \mapsto (x, v_x)$ $v_x \in T_x M$.

Proposition 9.4. *If v is a left invariant vector field on a Lie group G, then the time t flow is given by $g \mapsto g \exp(tx)$ where $x = v(1)$.*

Properties of the exponential map

Theorem 9.5 (thm 3.7 in Kirillov)**.** *The general exponential map statisfies the following:*

1) $\exp_*(0)\colon \mathfrak{g} \to T_1 G = \mathfrak{g}$ *is the identity map*

2) $\exp$ *is a diffeomorphism between some neighborhood of* 0 *in* $\mathfrak{g}$ *and some neighborhood of* 1 *in* G. *The inverse is denoted by* $\log$.

3) $\exp((t+s)x) = \exp(tx)\exp(sx)$ $\forall s, t \in \mathbb{K}$, $x \in \mathfrak{g}$

4) *If* $\varphi\colon G_1 \to G_2$ *is a morphism of Lie groups, then*

$$\exp(\varphi_*(x)) = \varphi(\exp(x)) \quad \forall x \in \mathfrak{g}_1 = T_1 G_1 \textit{(Recall: } \varphi_* = d\varphi(0)\textit{)}$$

5) *For* $X \in G, y \in \mathfrak{g}$ $X \exp(y) X^{-1} = \exp(\operatorname{Ad} X.y)$

Proof.
1) $\exp(x) = \gamma_x(1)$ for $x \in \mathfrak{g}$, and $\dot{\gamma}_x(0) = x$. So $(\exp_*(0))(x) = \dot{\gamma}_x(0) = x$.
2) Immediate by inverse function theorem and part 1).
3) Follows from the fact that $t \mapsto \exp(tx)$ is a one-parameter subgroup.
4) Follows from the uniqueness of the one-parameter subgroup. Let $x \in \mathfrak{g}_1$, consider $\dot{\gamma}\colon \mathbb{K} \to G_1 \to G_2$ by $t \mapsto \dot{\gamma}_x(t) \mapsto \varphi(\dot{\gamma}_x(t))$. This is a one-parameter subgroup of G_2. Hence $\dot{\gamma}(0) = \varphi_*(\dot{\gamma}_x(0)) = \varphi_*(x)$. So by uniqueness of a one-parameter subgroup of G_2 with $\dot{\gamma}(0) = \varphi_*(0)$. So $\gamma = \gamma_{\varphi_*(x)}$ in G_2.
5) Follows from 4) by $G \to G$ by $Y \mapsto XYX^{-1} = (\operatorname{Ad} X)(Y)$. ▫

10 January 31st, 2018

Classes of Manifolds

(1) *Complex manifold* $M \subseteq \mathbb{C}^n$: Coordinate systems $\varphi\colon U \to M$ where $U \subseteq \mathbb{C}^d$ are *holomorphic* functions.

(2) C^k*-manifolds* $M \subseteq \mathbb{R}^n$: Coordinate systems $\varphi\colon U \to M$ where $U \subseteq \mathbb{R}^d$, are differentiable of class C^k.

$$\begin{aligned} C^0 &= \text{ continuous functions} \\ C^1 &= \text{ 1st order differentiable functions} \\ C^k &= \text{ all partial derivatives up to order } k \text{ exist and are continuous} \\ C^\infty &= \text{ all partial derivatives exists } (\Rightarrow \text{continuous}) \\ &= \textit{ smooth} \text{ functions} \\ C^\omega &= \text{ all real analytic functions (functions with Taylor series expansions)} \end{aligned}$$

One could imagine a theory of C^k-Lie groups for $k \in \mathbb{N} \cup \{\infty, \omega\}$, in the real case. However we have:

Theorem 10.1 (Deep Theorem (see Remark 2.2))**.** *Let G be a real Lie group of class C^0. Then there is a Lie group G' of class C^ω such that $G \cong G'$ as C^0-Lie groups. Moreover, G' is unique up to isomorphism of of C^ω-Lie groups.*

The conclusion to be drawn from this is "every real Lie group is a C^ω-Lie group". it suffices to consider

– Complex Lie groups

– Real Lie groups

We may WLOG assume all maps involved have Taylor series expansions.

The Bracket (Commutator)

Let G be a Lie group and $\mathfrak{g} = T_1G$. Recall exp is a locally diffeomorphism.

$$\exp\colon \mathfrak{g} \to G$$

Let $U \subseteq \mathfrak{g}$ be a neighborhood of 0 and $V \subseteq G$ neighborhood of 1 such that $\exp|_U\colon U \to V$ is a diffeomorphism with an inverse denoted by log. consider the map

$$\varphi\colon \mathfrak{g} \times \mathfrak{g} \to G \quad \text{by} \quad (x, y) \mapsto \exp(x) \cdot \exp(y).$$

Then for $(x, y) \in \varphi^{-1}(\mu(x, y))$ for some $\mu(x, y) \in \mathfrak{g}$. Explicitly,

$$\mu(x, y) = \log(\exp(x)\exp(y))$$

definded for $(x, y) \in \varphi^{-1}(V) \subseteq \mathfrak{g} \times \mathfrak{g}$. μ is a real analytic (or holomorphic) function and thus has a Taylor series at $(0, 0)$.

Lemma 10.2. *$\mu(x, y) = x + y + \lambda(x, y) + \cdots$ where, $\lambda\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is a skew-symmetric bilinear map.*

Proof. General Taylor series:

$$f(\underline{x}) = \frac{1}{0!}L_0 + \frac{1}{1!}L_1(\underline{x}) + \frac{1}{2!}L_2(\underline{x}, \underline{x}) + \cdots$$

Where L_k is a multilinear function of k variables. In our case

$$\begin{aligned} \underline{x} &= (x, y) \in \mathfrak{g} \times \mathfrak{g} \cong \mathbb{R}^k \times \mathbb{R}^k \\ &= (x, 0) + (0, y). \end{aligned}$$

So

$$\mu(x, y) = c_0 + \alpha_1(x) + \alpha_2(y) + \frac{1}{2}(Q_1(x, x) + \lambda(x, y) + Q_2(y, y)) + \cdots$$

for some linear functions α_1, α_2 and bilinear functions Q_1, λ, Q_2. Observe

$$\mu(x, 0) = \log(\exp(x)\exp(0)) = x$$

So $c_0 = 0$, $\alpha_1(x) = x$, and $Q_1(x, x) = 0$. Similarly $\mu(0, y) = y$, which gives $\alpha_2(y) = y$ and $Q_2(y, y) = 0$. Lastly,

$$\mu(x, x) = \log(\exp(x)\exp(x)) = \log(\exp(2x)) = 2x.$$

So $\lambda(x, x) = 0\ \forall x \in \mathfrak{g}$ which implies $\lambda(x + y, x + y) = 0\ \forall x, y \in \mathfrak{g}$. Now since λ is bilinear we have, $\lambda(x, y) + \lambda(y, x) = 0$. □

Definition. The skew-symmetric bilinear function $\lambda\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ as introduced above is called the *commutator* (or *bracket*).

Notation. $[x, y] := \lambda(x, y)$.

Proposition 10.3.

1) *Let $\varphi\colon G_1 \to G_2$ be a morphism of Lie groups, and $\varphi_*\, \mathfrak{g}_1 \to \mathfrak{g}_2$ its differential. Then*

$$\varphi_*([x, y]) = [\varphi_*(x), \varphi_*(y)]$$

for any $x, y \in \mathfrak{g}_1$.

2) $\mathrm{Ad}_g([x, y]) = [\mathrm{Ad}_g(x), \mathrm{Ad}_g(y)]$ *for any $g \in G, x, y \in \mathfrak{g}$.*

3) *Let $X = \exp(x)$ and $Y = \exp(y)$. Then the* group commutator $[X, Y] = XYX^{-1}Y^{-1}$ *equals* $\exp([x, y] +$ *terms of higher order*).

Proof. 1) Recall for $x \in \mathfrak{g}_1$

$$\exp(\varphi_*(x)) = \varphi(\exp(x))$$

hence

$$\begin{aligned} \exp(\mu(\varphi_*(x), \varphi_*(y))) &= \exp(\varphi_*(x))\exp(\varphi_*(y)) \\ &= \varphi(\exp(x)\exp(y)) \\ &= \varphi(\exp(\mu(x, y))) \\ &= \exp(\varphi_*(\mu(x, y))). \end{aligned}$$

Now apply log to both sides.

2) $\mathrm{Ad}\, g\colon G \to G$ is a morphism of Lie groups. Apply 1).

3) Explicit calc. □

Corollary 10.4. *If G is a commutative Lie group then $[x, y] = 0$ for all $x, y \in \mathfrak{g}$.*

Proof. Use 3) from above. $\square$

Example 10.5. Let $G \subset GL(n, \mathbb{K})$ and $\mathfrak{g} \subset \mathfrak{gl}(n, \mathbb{K})$. Then,

$$\exp(x) = (1 + x + \cdots) \quad \exp(y) = (1 + y + \cdots)$$

This implies that

$$\exp(x)\exp(y)\exp(-x)\exp(y) = (1+x+\cdots)(1+y+\cdots)(1-x+\cdots)(1-y+\cdots) = 1+(xy-yx)+\cdots$$

So $[x, y] = xy - yx$.

11 February 2nd, 2018

Computing differentials using curves

Let $\varphi\colon M \to N$ be a morphism of manifolds. Let $p \in M$ and $x \in T_pM$. To find $d\varphi_p(x)$, also denoted $\varphi_*(x)$, let $\gamma\colon \mathbb{K} \to M$ be any morphism with $\gamma(0) = p$, and $d\gamma_0 = x^{\ddagger}$. Then $d(\varphi \circ \gamma)_0 = d\varphi_p \circ d\gamma_0 = d\varphi_p(x)$, where the first equality is by the chain rule. On the other hand

$$d(\varphi \circ \gamma)_0 = \frac{d}{dt}|_{t=0}\varphi(\gamma(t))$$

This is particularly useful for Lie groups: If $x \in \mathfrak{g}$ then $\gamma_x\colon \mathbb{K} \to G$, $\gamma_x(t) = \exp(tx)$ is a natural curve through $1 \in G$ with derivative x. So given a Lie group G, a manifold M and a morphism of manifolds $\varphi\colon G \to M$ its differential at 1 can be computed as follows:

$$\varphi_* = d\varphi_1\colon \mathfrak{g} \to T_\varphi(1)M \quad \varphi_*(x) = \frac{d}{dt}|_{t=0}\varphi(\exp(tx)).$$

Differential of Ad

Recall $\operatorname{Ad} g\colon G \to G$ by $x \mapsto gxg^{-1}$. Its differential is

$$\operatorname{Ad} g\colon \mathfrak{g} \to \mathfrak{g}.$$

So $\operatorname{Ad} g \in GL(\mathfrak{g})$, and $\operatorname{Ad}\colon G \to GL(\mathfrak{g})$. So its differential is $\operatorname{ad} = \operatorname{Ad}_*\colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$.

Lemma 11.1.

1) $\operatorname{ad} x.y = [x, y]$

2) $\operatorname{Ad}(\exp(X)) = \exp(\operatorname{ad} x)$

‡Here we identify $\operatorname{Hom}_{\mathbb{K}}(\mathbb{K}, T_pM)$ with T_pM

Proof. 1) For $g \in G$ consider $\operatorname{Ad} g \colon G \to G$. By formula for φ_* its differential is given by $\operatorname{Ad} g \colon \mathfrak{g} \to \mathfrak{g}$ by

$$(\operatorname{Ad} g)(y) = \frac{d}{dt}|_{t=0}(\operatorname{Ad} g)(\exp ty) = \frac{d}{dt}|_{t=0} g \exp(ty) g^{-1}$$

By the same formula again, $\operatorname{ad} \colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ is given by

$$\operatorname{ad} x.y = \frac{d}{ds}|_{s=0}\frac{d}{dt}|_{t=0} \exp(sx)\exp(ty)\exp(-sx) = \frac{d}{ds}|_{s=0}\frac{d}{dt}|_{t=0} \exp(ty+ts[x,y]+\cdots) = [x,y].$$

Where the second last equality is by Proposition 10.3 part 3).
2) Immediate by $X \exp(y) X^{-1} = \exp(\operatorname{Ad} X.y)$ which was proved earlier. ▫

Theorem 11.2 (Jacobi Identity)**.** *Let G be a Lie group and $\mathfrak{g} = T_1G$ Then the skew-symmetric bilinear map $[\cdot,\cdot] \colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfies the* Jacobi Identity*:*

$$[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0.$$

This identity can also be written (using skew-symmetry and bilinear-ity):

$$[x,[y,z]] = [[x,y],z] - [y,[x,z]]$$

$$\operatorname{ad} x.[y,z] = [\operatorname{ad} x.y, z] + [y, \operatorname{ad} x.z]$$

$$\operatorname{ad}[x,y] = \operatorname{ad} x \operatorname{ad} y = \operatorname{ad} y \operatorname{ad} x$$

Proof. Since $\operatorname{Ad} \colon G \to GL(\mathfrak{g})$ is a morphism of Lie groups, its differential preserves the commutator by Proposition 10.3 1). But the commutator on $\mathfrak{gl}(\mathfrak{g})$ is $[A,B] = AB - BA$ by example 10.5. So

$$\operatorname{ad}[x,y]_{\mathfrak{g}} = [\operatorname{ad} x, \operatorname{ad} y]_{\mathfrak{gl}(\mathfrak{g})} = \operatorname{ad} x \operatorname{ad} y - \operatorname{ad} y \operatorname{ad} x.$$

Applying both sides to $z \in \mathfrak{g}$ we get

$$[x,[y,z]] = [[x,y],z] - [y,[x,z]]$$

The other forms left as an exercise. ▫

12 February 5th, 2018

Lie algebras

Definition. A *Lie algebra* over a field $\mathbb{K}$ is a vector space together with a map $[\cdot,\cdot] \colon L \times L \to L$ satisfying

i) $[\cdot,\cdot]$ is bilinear

ii) $[x,x] = 0$ (skew-symmetric)[§]

[§] $\Rightarrow [x,y] = -[y,x]$ if $\operatorname{char} \mathbb{K} \neq 2$

iii) Jacobi identity holds.

Definition. A *homomorphism of Lie algebras* $\psi\colon \mathfrak{g}_1 \to \mathfrak{g}_2$ is a linear map such that $\psi([x,y]) = [\psi(x), \psi(y)]$.

Example 12.1. Let G be a Lie group. Then $\mathfrak{g} = T_1 G$ with the bracket $[x,y] = \lambda(x,y)$ as defined above is a Lie algebra. It is the *Lie algebra associated to* G and is sometimes denoted by $\mathrm{Lie}\,(G)$.

Every morphism $\varphi\colon G_1 \to G_2$ of Lie groups gives a homomorphism of Lie algebras $\varphi_*\colon \mathfrak{g}_1 \to \mathfrak{g}_2$. Moreover, $(\varphi \circ \psi)_* = \varphi_* \circ \psi_*$, $\mathrm{Id}_* = \mathrm{Id}$ (In other words, Lie is a functor from the category of Lie groups and Lie group morphisms to the category of Lie algebras and Lie algebra homomorphisms).

If G_1 is <u>connected</u> then

$$\mathrm{Hom}\,(G_1, G_2) \to \mathrm{Hom}\,(\mathfrak{g}_1, \mathfrak{g}_2)$$

is <u>injective</u> by $\varphi \mapsto \varphi_*$.

Example 12.2. Let A be any associative algebra over $\mathbb{K}$ (i.e. a ring containing $\mathbb{K}$). Then define

$$[a,b] = ab - ba.$$

This operation turns A into a Lie algebra.

Exercise 12.3. Show that $(A, [\cdot,\cdot])$ defined above is indeed a Lie algebra.

Definition. A Lie algebra $\mathfrak{g}$ is *abelian* if $[x,y] = 0$ for all $x, y \in \mathfrak{g}$.

Example 12.4. Any vector space L can be regarded as an abelian Lie algebra by defining $[x,y] = 0\ \forall x, y \in L$.

Subalgebras and Ideals

Definition. Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{K}$. A linear subspace $\mathfrak{h} \subset \mathfrak{g}$ is a

- *(Lie) subalgebra* if $[x,y] \in \mathfrak{h}$ for all $x, y \in \mathfrak{h}$.
- *(Lie) ideal* if $[x,y] \in \mathfrak{h}$ for all $x \in \mathfrak{g}, y \in \mathfrak{h}$.

Exercise 12.5. If $\mathfrak{h} \subset \mathfrak{g}$ is an ideal then $\mathfrak{g} / \mathfrak{h}$ is a Lie algebra with operation

$$[x + \mathfrak{h}, y + \mathfrak{h}] = [x,y] + \mathfrak{h}$$

for all $x, y \in \mathfrak{g}$.

Exercise 12.6. Show that $\mathfrak{sl}(n, \mathbb{K})$ is a subalgebra of $\mathfrak{gl}(n, \mathbb{K})$. (Recall $[x,y] = xy - yx$ in $\mathfrak{gl}(n, \mathbb{K})$ and $\mathfrak{sl}(n, \mathbb{K}) = \{A \in \mathfrak{gl}(n, \mathbb{K}) \mid \mathrm{Tr}\, A = 0\}$).

Theorem 12.7. *Let G be a Lie group and $\mathfrak{g} = \mathrm{Lie}\,(G)$.*

1. *If H is a Lie subgroup of G then $T_1 H$ is a Lie subalgebra in $\mathfrak{g}$.*
2. *If H is a normal closed subgroup of G then $\mathfrak{h} = T_1 H$ is a Lie ideal of $\mathfrak{g}$ and* $\mathrm{Lie}\,(G/H) \cong \mathfrak{g} / \mathfrak{h}$.

13 February 7th, 2018

Recall Theorem 12.7 from last lecture. We begin with a proof of this theorem.

Proof. (1) If $x \in T_1H$ then the one parameter subgroup $\gamma\colon \mathbb{K} \to H$ with $\dot{\gamma}(0) = x$ is also a one parameter subgroup of G and hence $\gamma(t) = \exp(tx)$ by uniqueness. So $\exp(tx) \in H$ $\forall t \in \mathbb{K}$. In particular for $x, y \in T_1H$:

$$\log(\exp(x)\exp(y)\exp(-x)\exp(-y))$$

belongs to $\mathfrak{h}$ hence by commutator formula, $[x, y] \in T_1H$. Similarly for part (2). (Read in book!) □

The Lie algebra of vector fields

M real manifold, $\mathrm{Diff}(M)$ be the group of diffeomorphisms. One can show that $\mathrm{Diff}(M)$ is an "∞-dimensional Lie group". What is "$\mathrm{Lie}\,(\mathrm{Diff}(M))$"?

Consider a one-parameter subgroup

$$\begin{aligned} \varphi\colon \mathbb{K} &\to \mathrm{Diff}(M) \\ t &\mapsto \varphi^t. \end{aligned}$$

So $\varphi^t\colon M \to M$ is a diffeomorphism $\forall t\mathbb{K}$. Then $\forall m \in M$, $t \mapsto \varphi^t(m)$ is a curve in M and thus

$$\frac{d}{dt}\varphi^t(m) \in T_mM.$$

Thus $\frac{d}{dt}\varphi^t$ is a vector field on M. So it is natural to define

$$\mathrm{Lie}\,(\mathrm{Diff}(M)) = \mathrm{Vect}\,(M)$$

the vector space of all smooth vector fields of M.

- Exponential map
- $\mathrm{Vect}\,(M)$ is a Lie algebra (usually ∞-dimensional) under commutator of vector fields:

$$[\sum f_i\partial_i, \sum g_j\partial_j] = \sum_{i,j}(g_j\delta_i(f_j) - f_i\partial_i(g_j))\partial_j$$

and

$$\partial_{[\xi,\,\zeta]}f = \partial_\zeta(\partial_\xi f) - \partial_\xi(\partial_\zeta f) \quad f \in C^\infty(M),\ \xi, \zeta \in \mathrm{Vect}\,(M)$$

Theorem 13.1. $\rho\colon G \to \mathrm{Diff}(M) \Rightarrow \rho_*\colon\ \mathfrak{g} \to \mathrm{Vect}\,(M)$ *Lie algebra homomorphism.*

Note. ρ_* is *action of* $\mathfrak{g}$ *by vector fields on* M

Example 13.2. $GL(n,\mathbb{R})\circlearrowright\mathbb{R}^n \rightsquigarrow \mathfrak{gl}(n,\mathbb{R}) \to \mathrm{Vect}\,(\mathbb{R}^n)\,,\ E_{ij} \mapsto x_j\partial_i$

$\mathfrak{g} \cong T_1 G \cong \operatorname{Vect}(G)^G \cong \{$1-parameter subgroup$\}$

G acts on itself by left multiplication.

$$\begin{aligned} &\mathrm{L}_g h = gh \quad L\colon G \to \operatorname{Diff}(G) \\ &\qquad\qquad\quad L_*\colon\ \mathfrak{g} \to \operatorname{Vect}(G) \\ &x \in \mathfrak{g} \rightsquigarrow \xi = L_* x \text{ is the right invariant vector field on } G \text{ such that } \xi(1) = x. \end{aligned}$$

Proof. $\exp(tx) \subset G$, $L_* x(g) = \frac{d}{dt}|_{t=0}(\exp(tx)g) = xg$. □

Theorem 13.3 (Corollarly 3.28 in Kirillov)**.** $\mathfrak{g} \cong \{$*right invariant vectorfields on* $G\}$ *is an isomorphism of Lie algebras.*

14 February 9th, 2018

Section 3.6: Read yourselves

Definition. Let $\mathfrak{g}$ be a Lie algebra. The *center* of $\mathfrak{g}$ is

$$\mathfrak{z}(\mathfrak{g}) = \{x \in \mathfrak{g} \mid [x, y] = 0 \quad \forall y \in \mathfrak{g}\}.$$

Exercise 14.1. $\mathfrak{z}(\mathfrak{g})$ is an ideal of $\mathfrak{g}$.

Theorem 14.2 (Thm 3.35 in Kirillov)**.** *Let G be a connected Lie group. Then its center*

$$Z(G) = \{g \in G \mid gh = hg \text{ for all } h \in G\}$$

is a closed Lie subgroup with Lie algebra $\mathfrak{z}(\mathfrak{g})$.

Baker-Campbell-Hausdorff formula

<u>Recall:</u> $\frac{1}{2}[x, y]$ $(x, y \in \mathfrak{g})$ is the quadratic term of the Taylor expansion of $\log(\exp(x)\exp(y))$ at 0.

Question 14.3. Do higher order terms give more information about? Or does the bracket completely determine the multiplication in G?

Theorem 14.4 (Baker-Campbell-Hausdorff)**.** *For small enough $x, y \in \mathfrak{g}$ we have,*

$$\exp(x)\exp(y) = \exp\Big(\sum_0^\infty \mu_n(x, y)\Big)$$

where

$$\begin{aligned} \mu_0(x, y) &= 0, \\ \mu_1(x, y) &= x + y, \\ \mu_2(x, y) &= \frac{1}{2}[x, y], \\ \mu_3(x, y) &= \frac{1}{12}\big([x[x, y]] + [y[y, x]]\big), \\ &\vdots \end{aligned}$$

In general, for $n \geqslant 0$ μ_n is a universal (=independent of $\mathfrak{g}$) expression, in a linear combination of commutators of degree n.

Corollary 14.5. *The group operation in a connected Lie group can be recovered from its Lie algebra.*

Section 3.8 Fundemental Theorems Study Carefully!

Given $\varphi\colon G_1 \to G_2$ a morphism of Lie groups, we get $\varphi_* \,\mathfrak{g}_1 \to \mathfrak{g}_2$ a Lie algebra homomorphism.

$$\begin{aligned} \mathrm{Hom}\,(G_1, G_2) &\to \mathrm{Hom}\,(\mathfrak{g}_1, \mathfrak{g}_2) \\ \varphi &\mapsto \varphi_* \end{aligned}$$

We have seen that if G_1 is connected that this map is injective. (i.e $\varphi_* = \psi_* \Rightarrow \varphi = \psi$). (When) Is $\varphi \mapsto \varphi_*$ surjective? I.e. (When) Can a Lie algebra homomorphism

$$f\colon\ \mathfrak{g}_1 \to \mathfrak{g}_2$$

be lifted to a Lie group homomorphism $G_1 \to G_2$?

Example 14.6. $G_1 = \mathbb{R}/\mathbb{Z} \cong S^1$, $G_2 = \mathbb{R}$, and $\mathfrak{g}_1 = \mathfrak{g}_2 = \mathbb{R}$ (abelian Lie algebra). Consider

$$\begin{aligned} &f\colon\ \mathfrak{g}_1 \to \mathfrak{g}_2 \\ &f(x) = x \quad \forall x \in \mathbb{R}. \end{aligned}$$

Then the corresponding $F\colon G_1 \to G_2$ would have to satisfy $F(x + \mathbb{Z}) = x$ for $x \in (0, 1)$. However,

$$F(\frac{1}{2} + \mathbb{Z} + \frac{1}{2} + \mathbb{Z}) = F(1 + \mathbb{Z}) = F(\mathbb{Z}) = 0,$$

and

$$F(\frac{1}{2} + \mathbb{Z}) + F(\frac{1}{2} + \mathbb{Z}) = \frac{1}{2} + \frac{1}{2} = 1.$$

Which is a contradiction.

Theorem 14.7 (Thm 3.40). *For any real or complex Lie group Gm there is a bijection between connected Lie subgroups $H \subset G$ and Lie subalgebras $\mathfrak{h} \subset \mathfrak{g}$ given by $H \mapsto \mathfrak{h} = \mathrm{Lie}\,(H) = T_1 H$.*

Proof. Skipped. □

Theorem 14.8 (Thm 3.41). *If G_1, G_2 are real or complex Lie groups and G_1 is connected and simply connected, then* $\mathrm{Hom}\,(G_1, G_2) \cong \mathrm{Hom}\,(\mathfrak{g}_1, \mathfrak{g}_2)$ *where* $\mathfrak{g}_i = \mathrm{Lie}\,(G_i)$.

Proof. Need to show that $\varphi \mapsto \varphi_*$ surjective. Consider the Lie group $G_1 \times G_2$ with Lie algebra $\mathfrak{g}_1 \times \mathfrak{g}_2$. The graph of $f\colon \mathfrak{g}_1 \to \mathfrak{g}_2$ is

$$\mathfrak{h} = \{(x, f(x)) \mid x \in \mathfrak{g}_1\} \subset \mathfrak{g}_1 \times \mathfrak{g}_2$$

is a Lie subalgebra. By theorem 14.7, $\mathfrak{h} = \mathrm{Lie}\,(H)$ for some connected Lie subgroup $H \subset G_1$.

$$H \hookrightarrow G_1 \times G_2 \xrightarrow{pr_1} G_1$$

gives morphism $\pi\colon H \to G_i$ with $\pi_*\colon \mathfrak{h} \to \mathfrak{g}_i$ and isomorphism. By exercises π is a covering map. G_1 simply connected, H connected implies π is an isomorphism. Now define φ by $G_1 \xrightarrow{\pi^{-1}} H \hookrightarrow G_1 \times G_2 \xrightarrow{pr_2} G_2$. φ morphism of Lie groups, $\varphi_* = f$. ▫

Theorem 14.9 (Thm 3.42 Lie's 3rd Thm)**.** *Any finite dimensional (real resp. complex) Lie algebra is isomorphic to a Lie algebra of a (real resp. complex) Lie group.*

Proof. Idea: Show that every Lie algebra is isomorphic to a subalgebra of $\mathfrak{gl}(n, \mathbb{K})$. (Ado's Theorem¶). Then use theorem 14.7. ▫

Corollary 14.10. *For any (real respectively complex) finite dimensional Lie algebra $\mathfrak{g}$ there is a unique up to isomorphism connected simply-connected (real resp. complex) Lie group G with* $\mathrm{Lie}\,(G) = \mathfrak{g}$*. Any other connected Lie group G' with Lie algebra $\mathfrak{g}$ is of the form G/Z for some discrete central subgroup $Z \subset G$.*

Corollary 14.11 (Cor 3.44)**.** *The category of finite dimensional Lie algebras is equivalent to the category of connected simply-connected Lie groups.*

$$\text{Conn. Simply-conn. Lie grps} \longrightarrow \text{f.d. Lie algs}$$

$$\begin{aligned} \text{Objects}&: \quad G \mapsto \mathrm{Lie}\,(G) \\ \text{Morphism}&: \quad \varphi\colon G_1 \to G_2 \mapsto \mathrm{Lie}\,(\varphi) = \varphi_*\colon \mathfrak{g}_1 \to \mathfrak{g}_2 \end{aligned}$$

Complex and Real Forms

Definition. The *complexification* of a real Lie algebra $\mathfrak{g}$ is

$$\mathfrak{g}_{\mathbb{C}} := \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$$

with bracket determined by

$$[x \otimes \lambda, y \otimes \mu] = [x, y] \otimes \lambda\mu \qquad \forall x, y \in \mathfrak{g},\ \lambda, \mu \in \mathbb{C}.$$

$\mathfrak{g}$ is a *real form* of $\mathfrak{g}_{\mathbb{C}}$.

¶If $\mathfrak{z}(\mathfrak{g}) = 0$, then $x \mapsto \mathrm{ad}\, x$ is injective.

Under the isomorphism $\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathfrak{g} \oplus i\, \mathfrak{g}$, we can also write it

$$[x+iy, z+iw] = [x,z] - [y,w] + i\big([y,z] + [x,w]\big) \qquad \forall x,y,z,w \in \mathfrak{g}\,.$$

Example 14.12. $\mathfrak{g} = \mathfrak{u}(n) \to \mathfrak{g}_{\mathbb{C}} = \mathfrak{gl}(n, \mathbb{C})$. With

$$X = \frac{1}{2}(X + X^*) + \frac{1}{2}(X - X^*)$$

with the first piece in $\mathfrak{u}(n)$ and the second piece in $i\mathfrak{u}(n)$. This is clear because X is hermitian $\Leftrightarrow$ iX skew-hermitian.

Definition. Let G be a connected complex Lie group, $\mathfrak{g} = \mathrm{Lie}\,(G)$. Let $K \subset G$ be a closed real Lie subgroup in G such that $\mathfrak{k} = \mathrm{Lie}\,(K)$ is a real form of $\mathfrak{g}$ (i.e. $\mathfrak{k} \otimes_{\mathbb{R}} \mathbb{C} = \mathfrak{g}$). Then K is a *real form* of G.

Example 14.13. $\mathfrak{su}(n)$ and $\mathfrak{sl}(n, \mathbb{R})$ are two (different) real forms of $\mathfrak{sl}(n, \mathbb{C})$, because

$$\mathfrak{sl}(n, \mathbb{C}) = \mathfrak{sl}(n, \mathbb{R}) \oplus i\mathfrak{sl}(n, \mathbb{R}) = \mathfrak{su}(n) \oplus i\mathfrak{su}(n).$$

SO $SU(n)$ and $SL(n, \mathbb{R})$ are two real forms of $SL(n, \mathbb{C})$.

Note. $SU(n) = SL(n, \mathbb{C}) \cap U(n)$. Where $SU(n)$ is a real Lie group, $SL(n, \mathbb{C})$ is a closed subset of $\mathbb{C}^{n^2} = \mathbb{R}^{2n^2}$ and $U(n)$ is a compact subset of $\mathbb{R}^{2n^2}$.

Exercise 14.14. Show that $SU(n)$ is a complact real form.

15 February 12th, 2018

Representations (=modules)

Definition. A *representation* of a Lie group is a finite dimensional vector space V together with a morphism $\rho = \rho_V \colon G \to GL(V)$ of Lie groups.

Definition. A*morphism* between two representation V and W of G is a linear map $f \colon V \to W$ which intertwines the action of G:

$$f \circ \rho_V(g) = \rho_W(g) \circ f$$

for all $g \in G$ i.e.
commutes.

Definition. V and W are *equivalent* (or *isomorphic*) if there exists invertible $f \colon V \to W$. Denoted $V \cong W$.

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d, "\rho_V(g)"'] & W \arrow[d, "\rho_W(g)"] \\ V \arrow[r, "f"] & W \end{tikzcd}$$

Definition. A *representation* of a Lie algebra $\mathfrak{g}$ is a vector space with a morphism $\rho\colon \mathfrak{g} \to \mathfrak{gl}(V)$ of Lie algebras.

A morphism of Lie algebra reps $f\colon V \to W$ is a defined as for groups. As are $\operatorname{Hom}_{\mathfrak{g}}(V, W)$ and $V \cong W$.

Note. We always assume that V is a <u>complex</u> vector space. If G is a real Lie group we regard $GL(V)$ as a real Lie group, and want $\rho\colon G \to GL(V)$ to be smooth.

Theorem 15.1. *G Lie group, $\mathfrak{g} = \operatorname{Lie}(G)$.*

(1) *Every representation $\rho\colon G \to GL(v)$ gives a representation $\rho_*\colon \mathfrak{gl} \to \mathfrak{gl}(V)$. Every morphism of G-representation is automatically a $\mathfrak{g}$-representation. In other words we have a functor*

$$\begin{aligned} \underline{\text{Rep}}G &\to \underline{\text{Rep}}\,\mathfrak{g} \\ (V, \rho) &\mapsto (V, \rho_*) \end{aligned}$$

(2) *If G is connected and simply connected, then the above is an <u>equivalence of categories</u>. In particular, any representation of $\mathfrak{g}$ can be lifted to a representation of G, and $\operatorname{Hom}_G(V, W) = \operatorname{Hom}_{\mathfrak{g}}(V, W)$.*

Example 15.2.

$$\rho\colon GL(2, \mathbb{C}) \to GL(\mathbb{C}[x, y]_d) \qquad A \mapsto \big(\rho(x, y) \mapsto \rho(ax + by, cx + dy)\big)$$

where $\mathbb{C}[x, y]_d = \bigoplus_{n=0}^{d} \mathbb{C}x^n y^{d-n}$. Also,

$$\rho_*\colon \mathfrak{gl}(2, \mathbb{C}) \to \mathfrak{gl}(\mathbb{C}[x, y]_d) \qquad E_{ij} \mapsto x_i \partial_j$$

with $(x_1, x_2) = (x, y)$.

Example 15.3.

$$\operatorname{Ad}\colon G \to GL(\mathfrak{g}) \qquad \operatorname{Ad}_* = \operatorname{ad}\colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}) \quad x \mapsto (\operatorname{ad} x\colon y \mapsto [x, y]).$$

Remark. If $\mathfrak{g}$ is a real Lie group then

$$\operatorname{Rep}\mathfrak{g} \cong \operatorname{Rep}\mathfrak{g}_{\mathbb{C}} \quad (V, \rho) \mapsto (V, \rho_{\mathbb{C}})$$

where

$$\rho_{\mathbb{C}}(x + iy) = \rho(x) + i\rho(y)$$

for any representation (V, ρ) of $\mathfrak{g}$ and

$$\operatorname{Hom}_{\mathfrak{g}_{\mathbb{C}}}(V, W) = \operatorname{Hom}_{\mathfrak{g}}(V, W).$$

Note. The theory of representations in real and/or ∞-dimensional vector spaces is very different.

16 February 14th, 2018

Operations on Representations

Let G be a Lie group, $\mathfrak{g}$ a Lie algebra.

Definition. Let V be a representation of G (respectively $\mathfrak{g}$). A *subrepresentation* of V is a linear subspace $W \subseteq V$ such that

$$\rho(x)W \subseteq W$$

for all $x \in G$ (respectively $x \in \mathfrak{g}$).

Definition. If V is a representation of G (or $\mathfrak{g}$), and $W \subseteq V$ is a subrepresentation then V/W becomes a representation:

$$\rho_{V/W}(x)(v + W) = \rho_V(x)v + W$$

for all $x \in G$ (or $\mathfrak{g}$) and $v + W \in V/W$.

Definition. For representations V and W we can define $V \oplus W$ by

$$\rho_{V\oplus W}(g)(v + w) = \rho_V(g)(v) + \rho_W(g)(w)$$

Definition. For a representation V we can define a representation structure on the dual space V^*. For Lie group case:

$$(\rho_{V^*}(g)\lambda)(v) = \lambda(\rho_V(g^{-1})v)$$

for all $v \in V$, $\lambda \in V^*$, $g \in G$. Lie algebra case:

$$(\rho_{V^*}(x)\lambda)(v) = \lambda(\rho_V(-x)v)$$

for $x \in \mathfrak{g}$.

Definition. For two representations V and W define a representation structure on $V \otimes W$. For the Lie group case:

$$\rho_{V\otimes W}(g)(v \otimes w) = \rho_V(g)v \otimes \rho_W(g)w$$

For Lie algebras: We compute $(\rho_{V\otimes W})_*$ to find the correct definition. Let $x \in \mathfrak{g}$. Consider $\gamma(t) = \exp(tx)$.

$$\begin{aligned}
(\rho_{V\otimes W})_*(x)(v \otimes w) &= \frac{d}{dt}|_{t=0}\rho_{V\otimes W}(\gamma(t))(v \otimes w) \\
&= \frac{d}{dt}|_{t=0}\rho_V(\gamma(t))v \otimes \rho_W(\gamma(t))w \\
&\overset{\text{Leibniz}}{=} \rho_V(\dot\gamma(0))v \otimes \rho_W(\gamma(0))w + \rho_V(\gamma(0))v \otimes \rho_W(\dot{\gamma(0)})w \\
&= \rho_V(x)v \otimes w + v \otimes \rho_W(x)w.
\end{aligned}$$

This motivates defining

$$\rho_{V\otimes W}(x)(v \otimes w) = \rho_V(x)v \otimes \rho_W(x)w.$$

Exercise 16.1. Check that $\rho_{V\otimes W}$ thus defined is indeed a representation of any Lie algebra $\mathfrak{g}$, given representations ρ_V, ρ_W.

Corollary 16.2. *If V is a representation of G (or $\mathfrak{g}$) then so is $V^{\otimes k} \otimes (V^*)^{\otimes \ell}$.*

Definition. Let V and W be representations. Then $\mathrm{Hom}\,(V, W) \cong V^* \otimes W$ by $(v \mapsto \lambda(v)w) \leftarrow\!\shortmid \lambda \otimes w$. This gives $\mathrm{Hom}\,(V, W)$ the structure of a representation with

$$(g.\varphi)(v) = g.\Big(\varphi(g^{-1}.v)\Big) \qquad g \in G$$

or

$$(x.\varphi)(v) = x.\Big(\varphi(v)\Big) - \varphi(x.v) \qquad x \in \mathfrak{g}$$

Invariants

Definition. A vector v in a representation V of G (or $\mathfrak{g}$) is *invariant* if $\rho(g)v = v\ \forall g \in G$ ($\rho(x)v = v\ \forall x \in \mathfrak{g}$). $V^G = \{v \in V \mid v \text{ is invariant}\}$ ($V^{\mathfrak{g}} = \{v \in V \mid v \text{ is invariant}\}$).

Example 16.3. $(\mathrm{Hom}\,(V, W))^G = \mathrm{Hom}_G\,(V, W)$ (respectively $(\mathrm{Hom}\,(V, W))^{\mathfrak{g}} = \mathrm{Hom}_{\mathfrak{g}}\,(V, W)$).

Example 16.4. B be a bilinear form on a representation V.

$$\begin{aligned} B\colon V \times V \to \mathbb{C} \text{ bilinear} &\Leftrightarrow B\colon V \otimes V \to \mathbb{C} \text{ linear} \\ &\Longleftrightarrow B \in (V \otimes V)^* \end{aligned}$$

So G (respectively $\mathfrak{g}$) acts on $(V \otimes V)^*$ via

$$(g.B)(v, w) = B(g^{-1}v, g^{-1}w)$$

respectively

$$(x.B)(v, w) = B(-x.v, w) + B(v, -x.w)$$

so B is invariant iff

$$B(v, w) = B(gv, gw) \qquad \forall g \in G$$

respectively

$$0 = B(x.v, w) + B(v, x.w) \qquad \forall x \in \mathfrak{g}\,.$$

17 February 16th, 2018

Irreducible Representations

Definition. A representaion $V \neq 0$ is *irreducible* (or *simple*) if the only subrepresentations of V are 0 and V. Otherwise V is *reducible.*

Example 17.1. The standard representation $\mathbb{C}^n$ of $SL(n, \mathbb{C})$ is irreducible. (Exercise) Hint: Use $I + E_{ij}$, $i \neq j$ to get $(1, 0, \cdots, 0)$.

Suppose $V \neq 0$ is reducible. Let $W \subset V$ be a proper nonzero subrepresentation. We get a SES

$$0 \to W \to V \to V/W \to 0 \quad (*)$$

Note. $\dim W$ and $\dim V/W$ are both strictly less than $\dim V$.

(When) does $(*)$ split? i.e $V \cong W \oplus V/W$?

Definition. A representation V is *completely reducible* (or *semisimple*) if

$$V \cong \oplus_{i=1}^{N} V_i, \quad V_i \text{ irreducible.}$$

Then:

$$V \cong \oplus_{i=1}^{k} n_i V_i = \oplus_{i=1}^{k} V_i^{\oplus n_i}$$

where $V_i \not\cong V_j$ $\forall i \neq j$. n_i is the *multiplicity* of V_i in V.

Example 17.2. $G = \mathbb{R}$, $\mathfrak{g} = \mathrm{Lie}\,(G) = \mathbb{R}$. A representation of $\mathfrak{g}$ is just a complex finite dimensional vector space V with an $\mathbb{R}$-linear map $\rho\colon \mathbb{R} \to \mathfrak{gl}(V) = End_{\mathbb{C}}(V)$. $\rho(t) = t\rho(1) = t \cdot A$, where $A = \rho(1)$, $A \in End_{\mathbb{C}}(V)$. Conversely any $A \in End_{\mathbb{C}}(V)$ gives a representation $\rho\colon \mathbb{R} \to \mathfrak{gl}(V)$. $V \cong W \Leftrightarrow A_V = TA_WT^{-1}$, some $T \in GL(V)$. This implies that Jordan canonical form classifies up to equivalence all representations of the Lie algebra $\mathbb{R}$. A representation of V is completely reducible iff Av is diagonalizable. V is irreducible iff $\dim V = 1$.

Some Goals of Representation Theory

1) Given G, classify irreducible representations of G.
2) Given a reducible representation, how to decompose it into irreducible representations?
3) For which G are <u>all</u> representations completely reducible?

Intertwining Operators (=morphisms of representation)

Suppose $A\colon V \to V$ is a diagonalizable intertwining operator:

$$V = \oplus_{\lambda\in\mathbb{C}} V_\lambda, \qquad V_\lambda = \{v \in V \mid Av = \lambda v\}$$

Then $\forall g \in G$, $\forall v \in V_\lambda$:

$$A\rho(g)v = \rho(g)Av = \rho(g)\lambda v = \lambda\rho(g)v.$$

This implies $\rho(g)v \in V_\lambda$, so $\rho(g)V_\lambda \subset V_\lambda$ for all $g \in G$. So $\forall\lambda$: V_λ is a subrepresentation of V.

Corollary 17.3. *If $z \in Z(G)$ such that $\rho(z)$ is diagonalizable, then $V = \oplus V_\lambda$ where V_λ= eigenspace of $\rho(z)$.*

Proof. $\rho(z)\rho(g) = \rho(zg) = \rho(gz) = \rho(g)\rho(z)$ $\forall g \in G$. This show that $\rho(z)$ is an intertwining operator. □

Example 17.4. $V = \mathbb{C}^n \otimes \mathbb{C}^n$ representations of $G = GL(n, \mathbb{C})$ $p\colon v \otimes w \mapsto w \otimes v$ commutes with G action implies it is an intertwiner. $V = V_+ \oplus V_-$ eigenspace decomposition.

$$V_+ = \mathrm{Span}\{v \otimes w + w \otimes v\}$$

$$V_- = \mathrm{Span}\{v \otimes w - w \otimes v\}$$

In face, $V_\pm$ are irreducible representations of G.

18 February 19th, 2018

Recall all representations are assumed complex and finite dimensional.

Schur's Lemma

Lemma 18.1 (Schur's Lemma)**.**

1) *Let V be an irreducible representation of G. Then*

$$\operatorname{Hom}_G(V, V) = \mathbb{C}\,\mathrm{Id}$$

2) *If V and W are irreducible representations, such that $V \ncong W$. Then* $\operatorname{Hom}_G(V, W) = 0$.

Proof. $\Phi\colon V \to W$ intertwining operator. Then $\ker\Phi$ and $\Im\Phi$ are subrepresentations of V and W respectively. V and W irreducible implies $\Phi = 0$ or an isomorphism. Which shows 2). To get 1), pick any eigenvalue λ of Φ. Then $\Phi - \lambda\,\mathrm{Id}$ is an intertwiner with $\ker(\Phi - \lambda\,\mathrm{Id}) \neq 0$. By V irreducible, we have $V = \ker(\Phi - \lambda\,\mathrm{Id})$. Thus $\Phi = \lambda\,\mathrm{Id}$. ▫

4.5 Unitary representations

<u>Goal:</u> Show that any representation of a compact real Lie group is completely reducible. Steps:

(1) Any *unitary* representation is completely reducible.

(2) Any representation of a compact real Lie group is unitary.

Definition. A representation (V, ρ) of G is *unitary* (or *unitarizable*) if there exists a positive definite Hermitian form on V, $(\cdot,\cdot)\colon V \times V \to \mathbb{C}$ that is G-invariant, i.e.

$$(g.u, g.v) = (u, v) \qquad \forall g \in G,\ u, v \in V.$$

A representation V of a Lie algebra $\mathfrak{g}$ is *unitarizable* if $\exists$ $(\cdot,\cdot)$ positive definite Hermitian form which is $\mathfrak{g}$-invariant:

$$(x.v, w) + (v, x.w) = 0 \qquad \forall x \in \mathfrak{g},\ v, w \in V.$$

Example 18.2. Let G be a finite group acting on a set X.

$$V = \mathbb{C}^X = \{\text{functions } f\colon X \to \mathbb{C}\}.$$

Define $\rho\colon G \to GL(V)$ by $(\rho(g)f)(x) = f^{-1}(g^{-1}.x)$ $x \in X$, $f \in V$, and $g \in G$. Then (V, ρ) is a rep of G. Define $(f, g) = \sum_{x \in X} f(X)\overline{g(x)}$, $f, g \in V$. Then $(\cdot,\cdot)$ is a G-invariant positive definite Hermitian form hence V is unitarizable.

Theorem 18.3. *Every unitarizable representation is completely reducible.*

Proof. The proof is by induction on $\dim V$. $W \subset V$ nonzero proper subrepresentation. Consider $W^\perp$ with respect to $(\cdot,\cdot)$ on V. We have $\forall v \in W^\perp,\ w \in W$

$$(g.v, w) \overset{*}{=} (v, g^{-1}.w) = 0.$$

Where $*$ is by G-invariance and $g^{-1}.w \in W$ because it is a subrepresentation. This implies that $g.W^\perp \subset W^\perp$, so $W^\perp$ is also a subrepresentation, $V = W \oplus W^\perp$. So $\dim W$, $\dim W^\perp < \dim V$. Proceed by induction. Same idea for for Lie algebras. ▫

Measure Theory on real Lie groups

Let $G \subseteq \mathbb{R}^n$ be a real Lie group. Recall $A \subseteq G$ is *open in* G (respectively. *closed in* G) if $A = B \cap G$ for some open (resp. closed) subset $B \subseteq \mathbb{R}^n$. Let $\Sigma \subseteq \mathcal{P}(G)$ be the smallest subset closed under complements and countable $\bigcup$, $\bigcap$, containing all open sets in G. [‖]

Definition. A *measure* on G is a map $\mu\colon \Sigma \to [0,\infty]$ such that

i) $\mu\left(\bigsqcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n)$ ($\bigsqcup$ denotes a disjoint union).

ii) $\mu(\varnothing) = 0$.

Definition.

$$\int_G c_i \chi_{A_i} d\mu = \sum c_i \mu(A_i).$$

Where $\chi_{A_i} = \begin{cases} 1 & x \in A_i \\ 0 & \text{otherwise} \end{cases}$. Through a limit process we can define $\int_G f d\mu$ *integral of f over G*.

4.6 Haar measure on compact Lie Groups

Definition. A *right Haar measure* on a real Lie group is a Borel measure dg which is invariant under right action of G on itself.

Thus if dg is a Haar measure on G then for an integrable function $f\colon G \to \mathbb{R}$ (i.e. $f \in L^1(G, dg)$)

$$\int_G f(gh)dg = \int_G f(g)dg \qquad \forall h \in G.$$

Example 18.4. Lebesgue measure on $\mathbb{R}$.

$$\int_{\mathbb{R}} f(x+y)dx = \int_{\mathbb{R}} f(x)dx \qquad \forall y \in \mathbb{R}.$$

[‖] Σ is called a σ-algebra.

Example 18.5. The Haar measure on $U(1)$ is given by $\frac{dz}{2\pi iz}$. We have:

$$\int_{U(1)} f(z)\frac{dz}{2\pi iz} = \begin{bmatrix} z = e^{2\pi i\theta} \\ dz = 2\pi iz d\theta \end{bmatrix} = \int_0^1 f(e^{2\pi i\theta})d\theta.$$

Note. $\forall w = e^{2\pi i\alpha} \in U(1)$ we have

$$\begin{aligned} \int_{U(1)} f(zw)\frac{dz}{2\pi iz} &= \int_0^1 f(e^{2\pi i(\theta+\alpha)})d\theta \\ &= \int_\alpha^{1+\alpha} f(e^{2\pi i\theta})d\theta \\ &= \int_\alpha^1 f(e^{2\pi i\theta})d\theta + \int_1^{1+\alpha} f(e^{2\pi i\theta})d\theta \\ &= \qquad\qquad\qquad + \int_0^\alpha f(e^{2\pi i\theta})d\theta \end{aligned}$$

by $\theta \mapsto \theta + 1$, so

$$= \int_0^1 f(e^{2\pi i\theta})d\theta.$$

Theorem 18.6. *Let G be a compact real Lie group. Then G has a canonical Borel measure dg which is invariant under*

$$\begin{aligned} &g \mapsto gh \qquad \forall h \in G \\ &g \mapsto hg \qquad \forall h \in G \\ &g \mapsto g^{-1} \end{aligned}$$

and such taht $\int_G dg = 1$. This is the Haar measure *on G.*

19 February 21st, 2018

We assume that G is a compact real Lie group.

Theorem 19.1. *Any (finite dimensional) representation of G is unitary, hence completely reducible.*

Proof. Let $(\cdot,\cdot)$ be any positive definite Hermitian form on a representation V of G. Define $\langle\cdot,\cdot\rangle\colon V \times V \to \mathbb{C}$ by

$$\langle v, w\rangle = \int_G (\rho(g)v, \rho(g)w)dg.$$

Then $\langle\cdot,\cdot\rangle$ is positive definite and Hermitian. Also $\forall h \in G$,

$$\langle \rho(h)v, \rho(h)w\rangle = \langle v, w\rangle$$

by left invariance of Haar measure. □

Characters

Definition. The *character* of a representation V of G is $\chi_V \colon G \to \mathbb{C}$, $\chi_V(g) = \mathrm{Tr}(\rho(g))$.

Lemma 19.2 (Lemma 4.4 in Kirillov)**.**

(1) $\chi_{\mathrm{triv}} = 1$ *(Recall:* triv $= \mathbb{C}$ *and* $g.1 = 1\ \forall g$.

(2) $\chi_{V \oplus W} = \chi_V + \chi_W$

(3) $\chi_{V \otimes W} = \chi_V \cdot \chi_W$

(4) $\chi_V(ghg^{-1}) = \chi_V(h) \qquad \forall g, h \in G$

(5) $\chi_{V^*}(g) = \overline{\chi_V(g)} \qquad \forall g \in G$.

Proof. Homework. ▫

Theorem 19.3 (Orthonormality of characters)**.**

(1) *If* V, W *are non-isomorphic irreducible representations, then*

$$\int_G \chi_V(g)\overline{\chi_W(g)}dg = 0.$$

(2) *If* V *is any irreducible representation*

$$\int_G |\chi_V(g)|^2 = 1.$$

Corollary 19.4. *If* $V \cong \oplus n_i V_i$ *where* V_i *are nonisomorphic irreducible representations, then* $n_i = (\chi_V, \chi_{V_i})\ \forall i$.

Corollary 19.5. *If* V *and* W *are two representations then* $V \cong W$ *iff* $\chi_V = \chi_W$

Proof. Homework. ▫

The Hilbert space $L^2(G, dg)$

Let G be a compact real Lie group. Let

$$L^2(G, dg) = \{f \colon G \to \mathbb{C} \mid \int_G |f(g)|^2 dg < \infty\}$$

This is a Hilbert space with respect to

$$(f_1, f_2) = \int_G f_1(g)\overline{f_2(g)}dg.$$

<u>Recall:</u> Hilbert $\Rightarrow$ Normed $\Rightarrow$ metric $\Rightarrow$ topological space. Hence we have a notion of closure and denseness.

G acts on the left and right:

$$\begin{aligned} (g.f)(h) &= f(hg) \qquad \forall g, h \in G \\ (f.g)(h) &= f(gh) \qquad \forall f \in L^2(G, dg). \end{aligned}$$

Matrix Coefficients

Let V be a representation of G. To $(\lambda, v) \in V^* \times V$ we associate a function $\rho_V^{\lambda,v} : G \to \mathbb{C}$ by $\rho_V^{\lambda,v}(g) = \lambda(\rho_V(g)v)$.

Definition. $\rho_V^{\lambda,v}$ is the *matrix coefficient* corresponding to V, λ, v.

Notes.

(1) $\rho_V^{\lambda,v} \in L^2(G, dg)$, because $\rho_V^{\lambda,v}$ is continuous and G is compact.

(2) The right hand side depends bilinearly on (λ, v), therefore we get a map $V^* \otimes V \to L^2(G, dg)$ by $x \mapsto (g \mapsto \rho_V^X(g))$.

(3) Under the isomorphism $V^* \otimes V \to End(V)$ this is equivalent to defining for $\varphi \in End(V)$,

$$\rho_V^{\varphi} : G \to \mathbb{C}, \qquad \rho_V^{\varphi}(g) = \operatorname{Tr}(\varphi \circ \rho_V(g))$$

Example 19.6. Fix a basis $\{v_i\}$ for V, let $v_i^* \in V$ be dual basis: $v_i^*(v_j) = \delta_{ij}$. Then $\rho_V^{v_i^* \otimes v_j}(g)$ is just the (i,j) entry of the matrix $\rho_V(g)$ in the basis $\{v_i\}$.

$$G \ni g \xrightarrow{\rho_V} \rho_V(g) \in GL(V) \cong GL(n, \mathbb{C}).$$

Taking $\mathbb{1} = \sum_i v_i^* \otimes v_i$, we get

$$\rho_V^{\mathbb{1}}(g) = \operatorname{Tr}(\rho_V(g)) = \chi_V(g).$$

So matrix coefficients generalize characters.

Theorem 19.7 (Orthogonality of Matrix Coefficients)**.** *Let G be a compact real Lie group.*

1) *Let $V \ncong W$ be irreducible representations. Then*

$$(\rho_V^{\varphi}, \rho_W^{\psi}) = 0$$

for all $\varphi \in End(V)$, $\psi \in End(W)$.

2) *Let V be an irreducible representation. Then*

$$(\rho_V^{\varphi_1}, \rho_V^{\varphi_2}) = \frac{\operatorname{Tr}(\varphi_1 \circ \varphi_2)}{\dim V}$$

for all $\varphi_1, \varphi_2 \in End(V)$.

Proof. For all irreducible representations V, W we have:

$$\begin{aligned}
(\rho_V^{\varphi_1}, \rho_W^{\psi}) &= \int_G \operatorname{Tr}(\varphi \circ \rho_V(g))\overline{\operatorname{Tr}(\psi \circ \rho_W(g))}dg \\
&= \operatorname{Tr}_{V \otimes W^*} \int_G (\rho_V(g) \circ \varphi) \otimes (\psi \circ \rho_W(g))^* dg \\
&= \operatorname{Tr}_{V \otimes W^*} \int_G (\rho_V(g) \circ \varphi) \otimes (\rho_{W^*}(g) \circ \psi^*) dg \\
&= \operatorname{Tr}_{V \otimes W^*} \int_G \rho_{V \otimes W^*}(g) \circ (\varphi \otimes \psi^*) dg \\
&= \operatorname{Tr}_{V \otimes W^*}(\Phi)
\end{aligned}$$

where $\Phi\colon V\otimes W^* \to V\otimes W^*$ is the value-average of $\varphi\otimes\psi$:

$$\Phi = \int_G \rho_{V\otimes W^*}(g)\circ(\varphi\otimes\psi^*)dg.$$

The image of Φ is thus contained in $(V\otimes W^*)^G$.

Now if $V \ncong W$ then

$$(V\otimes W^*)^G \cong \mathrm{Hom}_G(W,V) = 0$$

in which case we get $\Phi = 0$. Hence,

$$(\rho_V^\varphi, \rho_W^\psi) = \mathrm{Tr}\,\Phi = 0.$$

On the other hand if $W = V$ then

$$(V\otimes W^*)^G \cong End_G(V) = \mathbb{C}\cdot\mathrm{Id}_V\,.$$

Which means dim(range Φ) = 1. So

$$\Phi(\mathrm{Id}_v) = (\mathrm{Tr}\,\Phi)\cdot\mathrm{Id}_V \left(\Phi \sim \begin{bmatrix} \mathrm{Tr}\,\Phi & * & \cdots & * \\ 0 & & & \\ \vdots & & \text{\Large 0} & \\ 0 & & & \end{bmatrix}\right) \Rightarrow \mathrm{Tr}_V(\Phi(\mathrm{Id}_V)) = (\mathrm{Tr}\,\Phi)\cdot\dim V.$$

On the other hand,

$$\begin{aligned}
\mathrm{Tr}_V(\Phi(\mathrm{Id}_v)) &= \mathrm{Tr}_V \int_G \rho_{V\otimes V^*}(g)\circ(\varphi\otimes\psi^*)(\mathrm{Id}_V)dg \\
&= \mathrm{Tr}_V \int_G \rho_{V\otimes V^*}(g)\circ\varphi\circ\psi \\
&= \mathrm{Tr}_V \int_G \rho_V(g)\circ\varphi\circ\psi\circ\rho_V(g)^{-1}dg \\
&= \int_G \mathrm{Tr}_V(\rho_V(g)\circ\varphi\circ\psi\circ\rho_V(g)^{-1})dg \\
&= \int_G \mathrm{Tr}(\varphi\circ\psi)dg \\
&= \mathrm{Tr}(\varphi\circ\psi)
\end{aligned}$$

This implies that

$$(\rho_V^\varphi, \rho_V^\psi) = \mathrm{Tr}_{V\otimes V^*}(\Phi) = \frac{\mathrm{Tr}(\varphi\circ\psi)}{\dim V}$$

$\square$

20 February 23rd, 2018

Peter-Weyl Theorem

Let G be a compact real Lie group. The Peter-Weyl Theorem

1) describes $L^2(G, dg)$ as a G-bimodule

2) says that any $f \in L^2(G, dg)$ can be approximated by a linear combination of matrix coefficients coming from irreducible representations.

Let $\hat{G}$ be the set of equivalence classes of irreducible representations of G.

For $[V] \in \hat{G}$, define a G-invariant inner product on $V^* \otimes V \cong End(V)$

$$(\varphi, \psi) = \frac{\operatorname{Tr}(\varphi \circ \psi)}{\dim V}$$

Let $\hat{\bigoplus}_{[V]\in\hat{G}} V^* \otimes V$ denote the Hilbert space completion with respect to this form.

Theorem 20.1 (Peter-Weyl)**.** *The map*

$$m \colon \hat{\bigoplus}_{[V]\in\hat{G}} V^* \otimes V \to L^2(G, dg) \qquad \text{by} \qquad V^* \otimes V \ni X \mapsto \rho_V^X$$

is an isometric isomorphism of G-bimodules.

Proof. The onto part requires analysis, we skip the proof. ▫

Corollary 20.2. *The set of characters $\{\chi_V \mid [V] \in \hat{G}\}$ is an orthonormal Hilbert space basis for $L^2(G, dg)^G$, the space of conjugate-invariant functions on G.*

21 February 26th, 2018

Representations of $\mathfrak{sl}(2, \mathbb{C})$

Theorem 21.1. *Any (complex, finite-dimensional) representation of $\mathfrak{sl}(2, \mathbb{C})$ is completely reducible.*

Proof. Since $\mathfrak{sl}(2, \mathbb{C}) = \mathbb{C} \otimes_{\mathbb{R}} \mathfrak{su}(2)$, $\mathfrak{su}(2)$ and $\mathfrak{sl}(2, \mathbb{C})$ have the same complex representations.

Since $SU(2)$ is connected and simply connected, there is an equivalence of categories $\underline{\operatorname{Rep}}\, \mathfrak{su}(2) \cong \underline{\operatorname{Rep}}\, SU(2)$.

Since $SU(2)$ is a compact real Lie group every representation of $SU(2)$ is completely reducible. ▫

Note. There is also an algebraic proof of this result.

Recall $\mathfrak{sl}(2,\mathbb{C})$ has a basis $\{e,f,h\}$

$$e=\begin{bmatrix}0&1\\0&0\end{bmatrix},\qquad f=\begin{bmatrix}0&0\\1&0\end{bmatrix}, h=\begin{bmatrix}1&0\\0&-1\end{bmatrix}$$

with $[e,f]=h$, $[h,e]=2e$, $[h,f]=-2f$.

Theorem 21.2. *Let $\lambda\in\mathbb{Z}_{\geqslant 0}$, let $V_\lambda=\mathbb{C}[x,y]_\lambda=\mathbb{C}x^\lambda\oplus\mathbb{C}x^{\lambda-1}y\oplus\cdots\oplus\mathbb{C}y^\lambda$ and define*

$$\rho_\lambda\colon\mathfrak{sl}(2,\mathbb{C})\to\mathfrak{sl}(V_\lambda)$$

by

$$\rho_\lambda(e)=x\partial_y\qquad\rho_\lambda(f)=y\partial_x\qquad\rho_\lambda(h)=x\partial_x-y\partial_y$$

Then (V_λ,ρ_λ) is an irreducible representation of $\mathfrak{sl}(2,\mathbb{C})$.

Proof. We show that $[\rho_\lambda(e),\rho_\lambda(f)]=\rho_\lambda(h)$.

$$[\rho_\lambda(e),\rho_\lambda(f)]=[x\partial_y,y\partial_x]=x\partial_y\circ y\partial_x-y\partial_x\circ x\partial_y.$$

Apply to arbitrary $p\in V_\lambda$:

$$x\partial_y\left(y\frac{\partial p}{\partial x}\right)-y\partial_x\left(x\frac{\partial p}{\partial y}\right)=x\frac{\partial p}{\partial x}+xy\frac{\partial^2 p}{\partial y\partial x}-y\frac{\partial p}{\partial y}-yx\frac{\partial^2 p}{\partial x\partial y}=\rho_\lambda(h)(p).$$

Similarly $[\rho_\lambda(h),\rho_\lambda(e)]=2\rho_\lambda(e)$ and $[\rho_\lambda(h),\rho_\lambda(f)]=-2\rho_\lambda(f)$. Thus (V_λ,ρ_λ) is a representation of $\mathfrak{sl}(2,\mathbb{C})$.

Suppose $W\subseteq V_\lambda$ is a nonzero subrepresentation. Pick $P\in W$, $p\neq 0$.

$$\begin{aligned}\rho_\lambda(e)x^ky^\ell&=\ell\cdot x^{k+1}y^{\ell-1}\\ \rho_\lambda(f)x^ky^\ell&=k\cdot x^{k-1}y^{\ell+1}\\ \rho_\lambda(h)x^ky^\ell&=(k-\ell)\cdot x^ky^\ell\end{aligned}$$

Thus $\rho_\lambda(e)^kp\in\mathbb{C}^\times\cdot x^\lambda$ for appropriate $k\geqslant 0$, so $x^\lambda\in W$. Then $x^{\lambda-\ell}y^\ell\in\mathbb{C}^\times\cdot\rho_\lambda(f)^\ell x^\lambda\in W$ for $\ell=0,1,2,\dots,\lambda$. Thus $W=V_\lambda$. ▫

Note. **Note**. $\dim V_\lambda=\lambda+1$.

Theorem 21.3. *Any (finite dimensional complex) irreducible representation of $\mathfrak{sl}(2,\mathbb{C})$ is isomorphic to V_λ for some $\lambda\in\mathbb{Z}_{\geqslant 0}$.*

Proof. Let (V,ρ) be a finite dimensional complex irreducible representation of $\mathfrak{sl}(2,\mathbb{C})$. Put $(E,F,H)=(\rho(e),\rho(f),\rho(h))$ and for $\mu\in\mathbb{C}$ put $V[\mu]=\{v\in V\mid\rho(h)v=\mu v\}$**
Step 1:

$$\begin{aligned}EV[\mu]&\subseteq V[\mu+2]\\ FV[\mu]&\subseteq V[\mu-2]\\ HV[\mu]&\subseteq V[\mu]\end{aligned}$$

**This is called the *weight space*

hence, $V' := \bigoplus_{\mu\in\mathbb{C}} V[\mu]$ is a subrepresentation of V. Since V is finite dimensional and complex, $\exists\mu \in \mathbb{C}$ such that $V[\mu] \neq 0 \Rightarrow V' \neq 0 \Rightarrow V = V'$ since V is irreducible.

Step 2: $\exists\lambda \in \mathbb{C}$ and $v_\lambda \in V[\lambda]\backslash\{0\}$ with $Ev_\lambda = 0$. Indeed, pick any nonzero $w_\mu \in V[\mu]$ some $\mu \in \mathbb{C}$. Then $E^n w_\mu \in V[\mu + 2n]$. Since V is finite dimensional, eigenvectors are linearly independent, $\Rightarrow$ $\exists n \geqslant 0$ $E^n w_\mu \neq 0$, $E^{n+1}w_\mu = 0$. Put $\lambda = \mu + 2n$, $V_\lambda = E^n w_\mu$.

Step 3: $W_\lambda = \mathrm{span}\{F^n v_\lambda\}_{n\geqslant 0}$ is a submodule of V, hence $V = W_\lambda$. Indeed

$$\begin{aligned} HF^n v_\lambda &= (\lambda - 2n)F^n v_\lambda \\ FF^n v_\lambda &= F^{n+1} v_\lambda \\ EF^0 v_\lambda = Ev_\lambda &= 0 \end{aligned}$$

and for $n > 0$:

$$\begin{aligned} EF^n v_\lambda &= \big([E,F] + FE\big)F^{n-1}v_\lambda \\ &= HF^{n-1}v_\lambda + FEF^{n-1}V_\lambda. \end{aligned}$$

Now $HF^{n-1}v_\lambda \in W_\lambda$ and by induction $EF^{n-1}v_\lambda \in W_\lambda$. Thus $HF^{n-1}v_\lambda + FEF^{n-1}V_\lambda \in W_\lambda$.

Step 4: $\lambda \in \mathbb{Z}_{n\geqslant 0}$, $V = V[\lambda] \oplus V[\lambda - 2] \oplus \cdots \oplus V[-\lambda]$ and $V[\lambda - 2n] = \mathbb{C}F^n v_\lambda$.

Proof. Let n be the smallest positive integer such that $F^n v_\lambda = 0$. Then

$$\begin{aligned} 0 = EF^n v_\lambda &= [E, F^n]v_\lambda \\ &= (HF^{n-1} + FHF^{n-2} + \cdots + F^{n-1}H)v_\lambda \\ &= (\lambda - 2(n-1) + \lambda - 2(n-2) + \cdots + \lambda)F^{n-1}v_\lambda \end{aligned}$$

This implies $\lambda = n - 1 \Rightarrow n = \lambda + 1$ and $\{v_\lambda, Fv_\lambda, \ldots, F^\lambda v_\lambda\}$ is a basis. ▫

Step 5: Define

$$\Phi\colon V \to V_\lambda = \mathbb{C}[x,y]_\lambda$$

by $F^n v_\lambda \mapsto \frac{1}{(\lambda-n)!}x^{\lambda-n}y^n$. Then

$$\big(\Phi\colon \rho(f)\big)(F^n v_\lambda) = \Phi(F^{n+1}v_\lambda) = \frac{1}{(\lambda - n - 1)!}x^{\lambda-n-1}y^{n+1}$$

while

$$(\rho_\lambda \circ \Phi)(F^n v_\lambda) = y\partial_x(\frac{1}{(\lambda - n)!}x^{\lambda-n}y^n) \qquad = \frac{1}{(\lambda - n - 1)!}x^{\lambda-n-1}y^{n+1}.$$

Similarly $\Phi\circ\rho(e) = \rho_\lambda(e)\circ\Phi$ and $\Phi\circ\rho(h) = \rho_\lambda(h)\circ\Phi$. This implies a bijective intertwining operator. ▫

22 February 28th, 2018

Universal enveloping algebra

Important tool in representation theory.

Recall If A is an associative $\mathbb{K}$-algebra can turn A into a Lie algebra, $\mathcal{L}A$, by defining $[x,y] = xy - yx$.

Consider a representation $\rho\colon \mathfrak{g} \to \mathfrak{gl}(V)$.

Note. $\mathfrak{gl}(V) = \mathcal{L}End(V)$.

Replacing $End(V)$ by an arbitrary associative, we get the notion of an enveloping algebra.

Definition. An *enveloping algebra* of a Lie algebra $\mathfrak{g}$ is a pair (A, j) where A is an associative algebra and $j\colon \mathfrak{g} \to \mathcal{L}A$ is a Lie algebra morphism, i.e. $j([x,y]) = j(x)j(y) - j(y)j(x)$ and j is linear.

Any representation (V, ρ) gives an enveloping algebra $(End(V), \rho)$.

Question 22.1. What is the "most general" enveloping algebra?

Definition. Let $\mathfrak{g}$ be a Lie algebra over a field $\mathbb{K}$. The *universal enveloping algebra* of $\mathfrak{g}$, denoted $U\,\mathfrak{g}$ (or $U(\mathfrak{g})$), is the associative $\mathbb{K}$-algebra with 1 generated by symbols $i(x)$ for $x \in \mathfrak{g}$ subject to relations

$$\begin{aligned} i(x+y) &= i(x) + i(y) \qquad \forall x, y \in \mathfrak{g} \\ i(cx) &= ci(x) \qquad \forall x \in \mathfrak{g},\ c \in \mathbb{K} \end{aligned}$$

and

$$i([x,y]) = i(x)i(y) - i(y)i(x) \qquad \forall x, y \in \mathfrak{g}\,.$$

Remark. The map

$$\begin{aligned} \mathfrak{g} &\to U\,\mathfrak{g} \\ x &\mapsto i(x) \end{aligned}$$

which we might as well call i, is a Lie algebra morphism

$$i\colon \mathfrak{g} \to \mathcal{L}(U\,\mathfrak{g})$$

This is immediate by construction of $U\,\mathfrak{g}$: i is linear and $i([x,y]) = i(x)i(y) - i(y)i(x)$. Abusing notation, we write x instead of $i(x)$. This is ok since we will show $\mathfrak{g} \to U\,\mathfrak{g}$, $x \mapsto i(x)$ is injective.

Remark. $U\,\mathfrak{g} \cong T\,\mathfrak{g}/I$ where $I = (x \otimes y - y \otimes x - [x,y] \mid x, y \in \mathfrak{g})$ and $T\,\mathfrak{g} = \mathbb{K} \oplus \mathfrak{g} \oplus \mathfrak{g}^{\otimes 2} \oplus \mathfrak{g}^{\otimes 3} \oplus \cdots$. The isomorphism sends $i(x_1)\cdots i(x_n)$ to $x_1 \otimes \cdots \otimes x_n + I$.

Example 22.2. $\mathfrak{g}$ abelian Lie algebra with basis $\{x_i\}_{i=1}^n$. This implies $U\,\mathfrak{g} \cong S\,\mathfrak{g} \cong \mathbb{K}[x_1, \dots, x_n]$ a polynomial algebra. (Recall: $S\,\mathfrak{g} = T\,\mathfrak{g}/(x \otimes y - y \otimes x)$)

Example 22.3. $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$. Then $U\,\mathfrak{g}$ is the associative $\mathbb{C}$-algebra generated by e, f, h subject to the relations $ef - fe = h,\ he - eh = 2e,\ hf - fh = -2f$.

Note.

$$e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \in \mathfrak{sl}(2, \mathbb{C}) \subseteq M_2(\mathbb{C}).$$

In $M_2(\mathbb{C})$, $e^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ but in $U\,\mathfrak{g}$, $e^2 \neq 0$ (as we will see).

Example 22.4. The quadratic casimir element of $U(\mathfrak{sl}(2, \mathbb{C}))$ is $c = ef + fe + \frac{1}{2}h^2$. Then $c \in Z(U(\mathfrak{sl}(2, \mathbb{C})))$. Consider the following calculation

$$\begin{aligned} cf &= ef^2 + fef + \frac{1}{2}h^2 f \\ &= (fe + h)f + f(fe + h) + \frac{1}{2}h(fh - 2f) \\ &= fef + fh - 2f + ffe + fh + \frac{1}{2}(fh - 2f)(h - 2) \\ &= f(ef + fe) + f(h - 2 + h + \frac{1}{2}(h - 2)^2) \\ &= f(ef + fe + \frac{1}{2}h^2) = fc. \end{aligned}$$

Theorem 22.5 (Universal property of $U\,\mathfrak{g}$)**.** *Let (A, j) be any enveloping algebra of $\mathfrak{g}$. Then there exists a unique morphism $\tilde{j}\colon U\,\mathfrak{g} \to A$ of associative algebras such that $j = \tilde{j} \circ i$, i.e. the following diagram commutes*

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "i"] \arrow[dr, "j"'] & U\,\mathfrak{g} \arrow[d, dashed, "\exists!\tilde{j}"] \\ & A \end{tikzcd}$$

Corollary 22.6. *Given any representation*

$$\rho\colon\ \mathfrak{g} \to \mathfrak{gl}(V) = \mathcal{L}End(V)$$

there is a unique algebra morphism

$$\tilde{\rho}\colon U\,\mathfrak{g} \to End(V)$$

such that $\tilde{\rho}(i(x)) = \rho(x)\ \forall x \in \mathfrak{g}$.

This makes V into a $U\,\mathfrak{g}$-module: $X.v = \tilde{\rho}(X)v$. Conversely, given any $U\,\mathfrak{g}$-module V, we get a representation of $\mathfrak{g}$ by

$$\begin{gathered} \rho\colon\ \mathfrak{g} \to \mathfrak{gl}(V) \\ \rho(x)v = i(x)v \end{gathered}$$

So $\underline{\text{Rep}}(\mathfrak{g}) \cong U\,\mathfrak{g}\,-\underline{\text{Mod}}$ is an equivalence of categories.

23 March 2nd, 2018

Poincaré-Birkhoff-Witt Theorem

Theorem 23.1 (PBW Theorem). *$U(\mathfrak{g})$ is a filtered algebra and its associated graded algebra is congruent to $S(\mathfrak{g})$.*

Graded Algebras

Definition. A *gradation* $\mathcal{G}$ on an algebra A is a collection of subspaces $\mathcal{G}_n A$ (or just A_n if the gradation is obvious) such that

i) $A = \bigoplus_{n=0}^{\infty} A_n$;

ii) $A_n A_m \subseteq A_{n+m}$.

Example 23.2. The tensor algebra on a vector space is naturally graded

$$T(V) = \mathbb{K} \oplus V \oplus V^{\otimes 2} \oplus V^{\otimes 3} \oplus \cdots$$

namely $T(V)_n = \begin{cases} V^{\otimes n}, & n > 0 \\ \mathbb{K}, & n = 0 \end{cases}$.

Filtered Algebras

Definition. A *filtration* $\mathcal{F}$ on an algebra A is a collection of subspaces $\mathcal{F}_n A$ (or $A_{(n)}$ if the filtration is obvious) such that

i) $A_{(0)} \subseteq A_{(1)} \subseteq A_{(2)} \subseteq \cdots$;

ii) $\bigcup_{n=0}^{\infty} A_{(n)} = A$;

iii) $A_{(m)} A_{(n)} \subseteq A_{(m+n)}$.

Any graded algebra A is naturally filtered by $A_{(n)} = A_0 \oplus \cdots \oplus A_n$, but not conversely.

Example 23.3. $ef \in U(\mathfrak{sl}(2,\mathbb{C}))_{(2)}$ Note: $ef = fe + h$, so it is not graded. This filtration is uniquely determined by requiring $x \in U(\mathfrak{g})_{(1)}$ $\forall x \in \mathfrak{g}$.

Associated Graded Algebra

Think of this as formalizing the idea of a leading term.

Definition. Given a filtered algebra A the *associated graded algebra* is as follows

$$\text{gr } A = \bigoplus_{n=0}^{\infty} A_{(n)}/A_{(n-1)} \qquad A_{(-1)} = 0 \text{ by convention}$$

Example 23.4. $ef + U(\mathfrak{sl}(2,\mathbb{C}))_{(1)} = fe + U(\mathfrak{sl}(2,\mathbb{C}))_{(1)}$ in $(\text{gr } U(\mathfrak{sl}(2,\mathbb{C})))_2$.

So the PBW theorem says that gr $U(\mathfrak{g}) \cong S(\mathfrak{g})$ as graded algebras. As a word of warning, there is no algebra homomorphism from $A \to \text{gr } A$ as A is filtered.

However there is a function $f\colon A \to \text{gr } A$ defined as follows. Let $a \in A$ and $n \geqslant 0$ be the smallest such that $a \in A_{(n)}$. Then $f(a) = a + A_{(n-1)} \in (\text{gr } A)_n$. We do have that $f(ab) = f(a)f(b)$, but f is not linear.

Proof of PBW thm (sketch).
Step 1: Define $\varphi\colon S(\mathfrak{g}) \to \text{gr } (\mathfrak{g})$ by

$$\varphi(x_1 \cdots x_n) = x_1 \cdots x_n + U(\mathfrak{g})_{(n-1)}.$$

Is φ well-defined?

$$\varphi(x_1 \cdots x_i x_{i+1} \cdots x_n) - \varphi(x_1 \cdots x_{i+1} x_i \cdots x_n) = x_1 \cdots x_{i-1}[x_i, x_{i+1}]x_{i+2} \cdots x_n + U(\mathfrak{g})_{(n+1)} = 0$$

So we have that φ is well-defined. To show φ is onto, let $\{x_i\}_1^n$ be a basis.

$$\varphi\left(\sum_{k\in\mathbb{N}^n} c_k x_1^{k_1} \cdots x_n^{k_n}\right) = \sum_{\ell\in\mathbb{N}} \sum_{\substack{k\in\mathbb{N}^n \\ \sum k_i = \ell}} c_k x_1^{k_1} \cdots x_n^{k_n} + U(\mathfrak{g})_{(\ell-1)}$$

Using commutators we can reorder the terms, where $\varphi(x_1^{k_1} \cdots x_n^{k_n}) = x_1^{k_1} \cdots x_n^{k_n} + U(\mathfrak{g})_{(n-1)}$ are called *ordered monomials*. We want to show that any element $U(\mathfrak{g})$ can be written as a linear combination of ordered monomials.

It follows by induction in the following way,

$$xy = yx + [x, y]$$

were $x > y$. As an example

$$h^2 e = h(eh + 2e) = heh + 2he.$$

Finialy to show that φ is one-to-one, we show that there exists a unique way to reduce the following: for $x < y < z$ we want to reorder zyx. We can either switch $z \leftrightarrow y$ or $y \leftrightarrow x$. For the first choice we have

$$zyx = yzx + [z, y]x = yxz + y[z, x] + [z, y]x = xyz + [y, x]z + y[z, x] + [z, y]x, \qquad (*)$$

and for the later we have,

$$zyx = zxy + z[y, x] = xzy + [z, x]y + z[y, x] = xyz + x[z, y] + [z, x]y + z[y, x]. \qquad (**)$$

We need to show that the $(*) - (**) = 0$, but We notice that the remaining terms are exactly the Jacobian identity which is necessarily non-zero. □

24 March 5th, 2018

Structure Theory for Lie Algebras

The overall goal (in the field) is to classify all (finite dimensional) Lie algebras up to isomorphism. This is still an open problem. There two classes of Lie algebras that are the focus of this work

- Solvable
- Semi-simple

Main theorems

(1) Levi's theorem: any finite dimensional Lie algebra is a semi-direct product of a solvable and semi-simple Lie algebras

$$\mathfrak{g} = \mathfrak{g}_{\text{ss}} \ltimes \operatorname{rad}(\mathfrak{g})$$

where $\mathfrak{g}_{\text{ss}}$ is semi-simple, and $\operatorname{rad}(\mathfrak{g})$ is solvable.

(2) Classification of finite dimensional complex semi-simple Lie algebras (Root Systems)

The classification of solvable Lie algebras is still open (though possible non-interesting) Today we will focus on Solvable (and nilpotent) Lie algebras.

Commutant

Lemma 24.1 (1st isomorphism theorem)**.** *If $f\colon \mathfrak{g}_1 \to \mathfrak{g}_2$ is a morphism of Lie algebra, then* $\ker f$ *is an ideal of* $\mathfrak{g}_1$*,* $\operatorname{im} f$ *is a subalgebra of* $\mathfrak{g}_2$*, and* $\operatorname{im} f \cong \dfrac{\mathfrak{g}_1}{\ker f}$.

Lemma 24.2. *$I, J \subset \mathfrak{g}$ ideals. Then*

$$\begin{aligned} &I + J = \{x + y \mid x \in I, y \in J\} \\ &I \cap J \\ &[I, J] = \operatorname{span}\{[x, y] \mid x \in I, y \in J\} \end{aligned}$$

are ideals.

Definition. $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$ is the *commutant* (or *derived subalgebra*) of $\mathfrak{g}$.

Lemma 24.3.

(i) $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ *is abelian*

(ii) *If $I \subset \mathfrak{g}$ is an ideal such that $\mathfrak{g}/I$ is abelian, then $[\mathfrak{g}, \mathfrak{g}] \subseteq I$.*

Example 24.4. Let $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{K})$, then $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{sl}(n, \mathbb{K})$
($\subseteq$): This is clear as $\operatorname{Tr}(xy - yx) = 0$ for every $x, y \in \mathfrak{gl}(n, \mathbb{K})$.
($\supseteq$): For $i \neq j$ $E_{ii} - E_{jj} = [E_{ij}, E_{ji}] \in \mathfrak{g}'$ and $E_{ij} = \frac{1}{2}[E_{ii} - E_{jj}, E_{ij}] \in \mathfrak{g}'$

Derived Series

We construct the following series,

$$
\begin{aligned}
&D^0\,\mathfrak{g} = \mathfrak{g}\\
&D^1\,\mathfrak{g} = \mathfrak{g}' = [\mathfrak{g},\mathfrak{g}] = [D^0\,\mathfrak{g}, D^0\,\mathfrak{g}]\\
&D^2\,\mathfrak{g} = [\mathfrak{g}',\mathfrak{g}'] = [[\mathfrak{g},\mathfrak{g}],[\mathfrak{g},\mathfrak{g}]] = [\mathfrak{g}',\mathfrak{g}'] = [D^1\,\mathfrak{g}, D^1\,\mathfrak{g}]\\
&D^i\,\mathfrak{g} = [D^{i-1}\,\mathfrak{g}, D^{i-1}\,\mathfrak{g}]
\end{aligned}
$$

Notice that $\mathfrak{g} = D^0\,\mathfrak{g} \supseteq D^1\,\mathfrak{g} \supseteq D^2\,\mathfrak{g} \supseteq \cdots$.

Definition. $\mathfrak{g}$ is *solvable* if there exists $n > 0$: $D^n\,\mathfrak{g} = 0$.

Proposition 24.5. *Let $\mathfrak{g}$ be a Lie algebra, then TFAE:*

(i) $\mathfrak{g}$ *is solvable*

(ii) *There is a sequence of subspaces*

$$\mathfrak{g} = \mathfrak{a}^0 \supseteq \mathfrak{a}^1 \supseteq \mathfrak{a}^2 \supseteq \cdots \supseteq \mathfrak{a}^k = 0$$

such that $[\mathfrak{a}^i, \mathfrak{a}^{i+1}] \subseteq \mathfrak{a}^{i+1}$, $i = 0, \dots, k-1$ *and* $\mathfrak{a}^i/\mathfrak{a}^{i+1}$ *is abelian.*

Proof. (i) $\Rightarrow$ (ii): $\mathfrak{a}^i = D^i\,\mathfrak{g}$.
(ii) $\Rightarrow$ (i): $\mathfrak{a}^i/\mathfrak{a}^{i+1}$ is abelian applies $\mathfrak{a}^{i+1} \supseteq [\mathfrak{a}^i, \mathfrak{a}^i]$ for all i. We can use induction to show that $\mathfrak{a}^i \supseteq D^i\,\mathfrak{g}$ for all i. □

Lower Central Series

$D_0\,\mathfrak{g} = \mathfrak{g}$, $D_i\,\mathfrak{g} = [\mathfrak{g}, D_{i-1}\,\mathfrak{g}]$ for $i > 0$.

Definition. $\mathfrak{g}$ is *nilpotent* if there exists $n > 0$: $D_n\,\mathfrak{g} = 0$.

Proposition 24.6. *Let $\mathfrak{g}$ be a Lie algebra, then TFAE:*

(i) $\mathfrak{g}$ *is nilpotent*

(ii) *There is a sequence of subspaces*

$$\mathfrak{g} = \mathfrak{a}_0 \supseteq \mathfrak{a}_1 \supseteq \mathfrak{a}_2 \supseteq \cdots \supseteq \mathfrak{a}_k = 0$$

such that $[\mathfrak{g}, \mathfrak{a}_i] \subseteq \mathfrak{a}_{i+1}$, $i = 0, \dots, k-1$.

By induction it can be shown that $D^i\,\mathfrak{g} \subseteq D_i\,\mathfrak{g}$, so nilpotent implies solvable.

Example 24.7. $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{K})$. Consider,

$$
\begin{aligned}
\mathfrak{b} &= \{\text{upper triangular matrices}\} = \operatorname{span}_{\mathbb{K}}\{E_{ij} \mid i \leqslant j\}^{\dagger\dagger}\\
\mathfrak{n} &= \{\text{strictly upper triangular matrices}\} = \operatorname{span}_{\mathbb{K}}\{E_{ij} \mid i < j\}
\end{aligned}
$$

We claim that $\mathfrak{b}$ is solvable and $\mathfrak{n}$ is nilpotent.

†† This called the borel subalgebra

Proof of claim. We will instead prove a more general statement. Fix a flag

$$\mathcal{F} = (0 \subsetneq V_1 \subsetneq V_2 \subsetneq \cdots \subseteq V_k = V)$$

$V_i \subseteq V$ subspaces of a vector space V (finite dimensional). $\dim V_i \leqslant \dim V_{i+1}$; however, we do not require $\dim V_i = i$. We define,

$$\begin{aligned}
\mathfrak{b}(\mathcal{F}) &= \{x \in \mathfrak{gl}(V) \mid xV_i \subseteq V_i\ \forall i\} \\
\mathfrak{n}(\mathcal{F}) &= \{x \in \mathfrak{gl}(V) \mid xV_i \subseteq V_{i-1}\ \forall i\} \\
\mathfrak{a}_k(\mathcal{F}) &= \{x \in \mathfrak{gl}(V) \mid xV_i \subseteq V_{i-k}\ \forall i\}
\end{aligned}$$

with $V_k = 0$ for $k < 0$. (If $\mathcal{F} = \mathcal{F}_{\text{std}}$, then $V := \operatorname{span}\{\mathbb{e}_1, \ldots, \mathbb{e}_n\}$ $\mathfrak{b}(\mathcal{F}_{\text{std}}) = \mathfrak{b}$, $\mathfrak{n}(\mathcal{F}_{\text{std}}) = \mathfrak{n}$. Obviously, $\mathfrak{a}_k \cdot \mathfrak{a}_\ell \subseteq \mathfrak{a}_{k+\ell}$ and $[\mathfrak{a}_k, \mathfrak{a}_\ell] \subseteq \mathfrak{a}_{k+\ell}$, hence $\mathfrak{n}(\mathcal{F})$ is nilpotent. Since diagonal entries of xy and yx coincide for $x, y \in \mathfrak{b}$ (check!). We have $D^1\mathfrak{b} \subseteq \mathfrak{a}_1$, and by induction we can see that $D^i\mathfrak{b} \subseteq \mathfrak{a}_{2^i}$. This implies that $\mathfrak{b}$ is solvable. ▫

25 March 7th, 2018

Lie's Theorem

Theorem 25.1 (Lie's Theorem)**.** *Let $\rho\colon \mathfrak{g} \to \mathfrak{gl}(V)$ be a complex finite dimensional representation of a solvable Lie algebra (real or complex), then there is a basis for V in which all $\rho(x)$, $x \in \mathfrak{g}$ is upper-triangular.*

Proof. Step 1: We show $\exists v \in V$ which is a common eigenvector for all $\rho(x)$, $x \in \mathfrak{g}$ by induction on $\dim \mathfrak{g}$. $\dim \mathfrak{g} = 1$ is obvious. $\dim \mathfrak{g} > 1$ since $\mathfrak{g}$ is solvable $[\mathfrak{g}, \mathfrak{g}] \subsetneq \mathfrak{g}$. Chose subspace $\mathfrak{g}'$, $[\mathfrak{g}, \mathfrak{g}] \subseteq \mathfrak{g}' \subseteq \mathfrak{g}$ such that $\dim \mathfrak{g}' = \dim \mathfrak{g} - 1$. Then $\mathfrak{g} = \mathfrak{g}' \oplus \mathbb{C}x$ for some $x \in \mathfrak{g}$ and $\mathfrak{g}'$ is an ideal of $\mathfrak{g}$:

$$[\mathfrak{g}, \mathfrak{g}'] \subseteq [\mathfrak{g}, \mathfrak{g}] \subseteq \mathfrak{g}'.$$

$\mathfrak{g}$ solvable implies $\mathfrak{g}'$ is solvable. By induction hypothesis, we can find an eigenvector $v \in V$ for $\rho(h)$, for all $h \in \mathfrak{g}'$.

Let $W = \operatorname{span}\{v^0 = v,\ v^1 = \rho(x)v,\ v^2 = \rho(x)^2 v, \ldots\}$. $\dim W \leqslant \dim V < \infty \Rightarrow\ \exists n > 0$ such that $v^{n+1} = \rho(x^{n+1})v \in \operatorname{span}\{v^0, \ldots, v^n\}$.

Claim: W is a subrepresentation of V.

$$\begin{aligned}
\rho(h)v^k &= \rho(h)\rho(x)^k v \\
&= \rho(x)^k \rho(h) v + [\rho(h), \rho(x)^k] v \\
&= \lambda(h)\rho(x)^k v + \operatorname{span}\{v^{k-1}, \ldots, v^0\}
\end{aligned}$$

So $\rho(h)$ is upper triangular on W. We can show that $\mathfrak{g}'$ acts diagonally on W with $\rho(h)v^k = \lambda(h)v^k$. This implies that we can take $w \in W$ as the common eigenvector for $\rho(x)$.

Step 2: We proceed by induction on $\dim V$. For $\rho\colon\ \mathfrak{g} \to \mathfrak{gl}(V)$, by the previous step, we get a common eigenvector $x \in V$ of $\rho(y)$, $y \in \mathfrak{g}$. Then $\mathbb{C}x$ is a subrepresentation of V. $\dim V/\dim \mathbb{C}x < \dim V$, by induction there exists basis $\{v_1, \ldots, v_n\}$ for $V/\mathbb{C}x$ such that

$$\begin{aligned} \rho_{V/\mathbb{C}x}\colon\ \mathfrak{g} &\to \mathfrak{gl}(V/\mathbb{C}x) \\ y &\mapsto (w + \mathbb{C}y \mapsto \rho(y)w + \mathbb{C}x) \end{aligned}$$

has all images $\rho_{V/\mathbb{C}x}(y)$ upper-triangular.

Pick $\tilde{v}_i \in V$ such that $v_i = \tilde{v}_i + \mathbb{C}x$, then $\rho(y)$ is upper-triangular $\forall y \in \mathfrak{g}$.

$$\rho(y) = \begin{array}{c c} & \begin{array}{cccc} x & \tilde{v}_1 & \cdots & \tilde{v}_n \end{array} \\ \begin{array}{c} x \\ \tilde{v}_1 \\ \vdots \\ \tilde{v}_n \end{array} & \begin{bmatrix} * & \cdots & \cdots & * \\ & * & \cdots & \vdots \\ & & \ddots & \vdots \\ 0 & & & * \end{bmatrix} \end{array}$$

□

Corollary 25.2. *Any complex finite dimensional irreducible representation of a solvable Lie algebra is 1-dimensional.*

Engel's Theorem

Theorem 25.3 (Engel's Theorem)**.** *Let V be a finite dimensional real or complex vector space. Let $\mathfrak{g} \subseteq \mathfrak{gl}(V)$ be a Lie subalgebra consisting of nilpotent operators. Then there exists a basis for V in which all $x \in \mathfrak{g}$ are strictly upper-triangular.*

Corollary 25.4. *If $\mathfrak{g}$ is a finite dimensional Lie algebra, the $\mathfrak{g}$ is nilpotent iff $\forall x \in \mathfrak{g}$, the map $\operatorname{ad} x\colon\ \mathfrak{g} \to \mathfrak{g}$ is nilpotent.*

26 March 9th, 2018

Lie's Theorem Again

Theorem 26.1 (Lie Theorem again)**.** *Any complex finite dimensional representation of a solvable Lie algebra has a flag $V = V_n \supsetneq V_{n-1} \supsetneq \cdots \supsetneq V_0 = 0$ which*

$$\begin{aligned} &\underline{\mathfrak{g}}\text{-}stable: \ x.V_i \subseteq V_i \quad \forall x \in \mathfrak{g} \\ &and\ \underline{complete}: \ \dim V_i = i \end{aligned}$$

Proof. (1) If W_1 is a representation and $W_2 \subseteq W_1$ is a subrepresentation such that W_2 and W_1/W_2 have $\mathfrak{g}$-stable complete flags, then so does W_1. (Exercise)

(2) If $\mathfrak{g}$ is abelian, then the claim is true: Pick any common eigenvector $v \in V$ for all of $\mathfrak{g}$. By induction, $V/\mathbb{C}v$ has a $\mathfrak{g}$-stable complete flag and $\mathbb{C}v \supsetneq 0$ is one for $\mathbb{C}$. By (1), V does.

(3) If $\mathfrak{g}$ is stable, consider $W = [\mathfrak{g},\mathfrak{g}]V$. This is a proper submodule. Consider V/W. This is a finite dimensional representation of $\mathfrak{a} = \mathfrak{g}/[\mathfrak{g},\mathfrak{g}]$ which is abelian Lie algebra. By (2) V/W has an $\mathfrak{a}$-stable complete flag. Then that flag is $\mathfrak{g}$-stable. By induction, W has a $\mathfrak{g}$-stable complete flag, so by (1) so does V. □

The Radical; semi-simple and reductive Lie algebras

Definition. A Lie algebra $\mathfrak{g}$ is called *semi-simple* if $\{0\}$ is the only solvable ideal in $\mathfrak{g}$.

Definition. A Lie algebra $\mathfrak{g}$ is called *simple* if it is non-abelian and $\{0\}$ and $\mathfrak{g}$ are the only ideals.

Lemma 26.2. $\mathfrak{g}$ *simple* $\Rightarrow$ $\mathfrak{g}$ *semi-simple.*

Proof. Let $\{0\} \neq I \subseteq \mathfrak{g}$ be a solvable ideal. $\mathfrak{g}$ simple $\Rightarrow$ $I = \mathfrak{g}$ $\Rightarrow$ the ideal $[\mathfrak{g},\mathfrak{g}]$ is proper $\Rightarrow$ $[\mathfrak{g},\mathfrak{g}] = 0$ $\Rightarrow$ $\mathfrak{g}$ abelian, which is a contradiction. □

Example 26.3. $\mathfrak{sl}(2,\mathbb{C})$ is simple. Use the map $\operatorname{ad} h$.

Example 26.4. In any finite dimensional Lie algebra $\mathfrak{g}$, $\exists!$ maximal solvable ideal, called the *radical*, denoted $\operatorname{rad}(\mathfrak{g})$.

Proof. Let $\operatorname{rad}(\mathfrak{g})$ be the sum of all solvable ideals of $\mathfrak{g}$.

Claim: $\operatorname{rad}(\mathfrak{g})$ is solvable.

Proof of claim. If I and J are solvable ideals of $\mathfrak{g}$, then $\dfrac{I+J}{I} = \dfrac{J}{I\cap J}$. Since J is solvable, $\dfrac{J}{I\cap J}$ is solvable. Since I and $\dfrac{I+J}{I}$ are solvable $\Rightarrow$ $I+J$ solvable. By induction, any finite sum is solvable. $\mathfrak{g}$ finite dimensional implies there exists $N > 0$ such that $D^N I = 0$ for all solvable ideals of $\mathfrak{g}$ ($N = \dim\mathfrak{g}$). So we show that $D^N \operatorname{rad}(\mathfrak{g}) = 0$. $\{x_i\}_{i=1}^{2^N} \subset \operatorname{rad}(\mathfrak{g})$, then $\{x_i\}_{i=1}^{2^N} \subset \sum_{j=1}^{M} I_j$ for some finitely many solvable ideals. Hence, $D^N \operatorname{rad}(\mathfrak{g}) = 0$. □

It is clear by the construction of $\operatorname{rad}(\mathfrak{g})$ that it is the unique maximal solvable ideal of $\mathfrak{g}$. □

Theorem 26.5. $\operatorname{rad}(\mathfrak{g}/\operatorname{rad}(\mathfrak{g})) = 0$*, i.e.* $\mathfrak{g}/\operatorname{rad}(\mathfrak{g})$ *is semi-simple. (Note* $\mathfrak{g}$ *is semi-simple iff* $\operatorname{rad}(\mathfrak{g}) = 0$*)*

Corollary 26.6. *The following*

$$0 \to \operatorname{rad}(\mathfrak{g}) \to \mathfrak{g} \to \mathfrak{g}_{\mathrm{ss}} \to 0$$

is an exact sequence , i.e. $\mathfrak{g}_{\mathrm{ss}} = \mathfrak{g}/\operatorname{rad}(\mathfrak{g})$.

Question 26.7. When is there and inclusion $\mathfrak{g}_{\mathrm{ss}} \hookrightarrow \mathfrak{g}$ such that when composed with the surjective map from $\mathfrak{g} \to \mathfrak{g}_{\mathrm{ss}}$ from the previous corollary, we get $\mathrm{Id}_{\mathfrak{g}_{\mathrm{ss}}}$?

The answer to this question is the following theorem.

Theorem 26.8 (Levi's Theorem). *Any finite dimensional Lie algebra* $\mathfrak{g}$ *can be decomposed as a vector space*

$$\mathfrak{g} = \mathfrak{g}_{\text{ss}} \oplus \operatorname{rad}(\mathfrak{g})$$

where $\mathfrak{g}_{\text{ss}}$ *is semi-simple is a subalgebra isomorphic to* $\mathfrak{g} / \operatorname{rad}(\mathfrak{g})$.

(Semi) direct products of Lie algebras

Internal case

Definition. $\mathfrak{g}$ is the *internal direct product* of two subalgebras I_1 and I_2 If the following hold:

(1) $\mathfrak{g} = I_1 \oplus I_2$ as vector spaces
(2) I_1 and I_2 are ideals.

Notation: $\mathfrak{g} = I_1 \times I_2$.

Definition. $\mathfrak{g}$ is the *(internal) semi-direct product* of two subalgebras I_1 and I_2 If the following hold:

(1) $\mathfrak{g} = I_1 \oplus I_2$ as vector spaces
(2) I_1 or I_2 is an ideal.

Notation: $\mathfrak{g} = I_1 \ltimes I_2$ if (I_2 is the ideal).

Example 26.9. I_2 is an ideal

$$[x_1 + x_2, y_1 + y_2] = [x_1, y_1] + ([x_1, y_2] + [x_2, y_1] + [x_2, y_2])$$

Where $[x_1, y_1] \in I_1$ and the terms in parenthesis is in I_2 because I_2 is an ideal.

External

Definition. Let $\mathfrak{g}_1$ and $\mathfrak{g}_2$ be Lie algebras, and $\alpha \colon \mathfrak{g}_1 \to \operatorname{Der}(\mathfrak{g}_2)$ be a Lie algebra morphism. Define $\mathfrak{g}_1 \ltimes_\alpha \mathfrak{g}_2 = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ as vector spaces, and

$$[(x_1, y_1), (x_2, y_2)] = ([x_1, x_2] + \alpha(y_1)(x_2) - \alpha(x_2)(y_1), [y_1, y_2]) \,.$$

27 March 19th, 2018

This lecture will focus on a easy method for checking whether or not a give Lie algebra is semi-simple based on the Killing form.

Bilinear forms

Definition.

a) Given a representation $\rho\colon \mathfrak{g} \to End(V)$ define a bilinear form $b_{\rho,V}\colon \mathfrak{g}\times\mathfrak{g} \to \mathbb{K}$ by

$$b_{\rho,V} = \mathrm{Tr}(\rho(x)\rho(y))$$

called the *trace form*.

b) In particular, for

$$\begin{aligned} \mathrm{ad}\colon \mathfrak{g} &\to End(\mathfrak{g}) \\ x &\mapsto \mathrm{ad}\, x = [x, -] \end{aligned}$$

$\kappa(x, y) = \rho_{\mathrm{ad}\,\mathfrak{g}} = \mathrm{Tr}(\mathrm{ad}(x)\,\mathrm{ad}(y))$ called the *Killing form*.

Goal: Is to get to the Cartan's criterion

a) $\mathfrak{g}$ is a solvable iff $\kappa(\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]) = 0$.

b) $\mathfrak{g}$ is semi-simple iff κ is non-degenerate.

Example 27.1.

a) $\mathfrak{g} = \mathfrak{gl}(2, \mathbb{C}) \curvearrowright \mathbb{C}^2$

$$\begin{aligned} \rho\colon \mathfrak{gl}(2,\mathbb{C}) &\to End(\mathbb{C}^2) \\ e &\mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ f &\mapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \\ h &\mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ I &\mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

If we check $b_{\rho,\mathbb{C}^2}(e, e) = \mathrm{Tr}\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) = 0$. Checking for the other basis elements we can see that $b_{\rho.\mathbb{C}^2}$ can be represented by

$$B = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

with respect to the ordered basis (e, f, h, I).

b) $\mathrm{ad}\colon \mathfrak{gl}(2,\mathbb{C}) \to End(\mathfrak{gl}(2,\mathbb{C}))$. By the same process before we can see that κ represented by

$$\begin{pmatrix} 0 & 4 & 0 & 0 \\ 4 & 0 & 0 & 0 \\ 0 & 0 & 8 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Recall a bilinear form $b\colon \mathfrak{g} \times \mathfrak{g} \to \mathbb{K}$ is:

- invariant if

$$b(\mathrm{ad}(x)(y), z) + b(y, \mathrm{ad}(x)(z)) = 0 (\Leftrightarrow b([y,x], z) = b(y, [x,z]))$$

- non-degenerate if $b(x, -) \neq 0$ for all $x \in \mathfrak{g}$, i.e. $\forall x \in \mathfrak{g}\ \exists y \in \mathfrak{g}$ such that $b(x,y) \neq 0$.

Lemma 27.2. *$b\colon \mathfrak{g} \times \mathfrak{g} \to \mathbb{K}$ invariant, $I \subset \mathfrak{g}$ ideal, implies that*

$$I^{\perp} = \{x \in \mathfrak{g} \mid b(x,z) = 0\ \forall z \in I\}$$

is an ideal in $\mathfrak{g}$.

Proof. $x \in I^{\perp}$, we show that $[x,y] \in I^{\perp}$ for all $y \in \mathfrak{g}$.

$$b([x,y], z) = b(x, [y,z]) = 0$$

hence, $[x,y] \in I^{\perp}$. $\square$

Corollary 27.3. $\mathfrak{g}^{\perp} = \ker(b)$ *is an ideal in $\mathfrak{g}$.*

Lemma 27.4. *Given $\rho\colon \mathfrak{g} \to End(V)$ $b_{\rho,V}$ is a symmetric invariant bilinear form.*

Proof. Consider the following calculation

$$\begin{aligned} b([x,y], z) + b(y, [x,z]) &= \mathrm{Tr}(\rho([x,y])\rho(z)) + \mathrm{Tr}(\rho(x)\rho([x,y])) \\ &= \mathrm{Tr}(\rho(x)\rho(y)\rho(z) - \rho(y)\rho(x)\rho(z) + \rho(y)\rho(x)\rho(z) - \rho(y)\rho(z)\rho(x)) \\ &= 0. \end{aligned}$$

Where the last equality is by the additivity of trace and $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$, which also symmetry. $\square$

Theorem 27.5. *If $\exists \rho\colon \mathfrak{g} \to End(V)$ such that $b_{\rho,V}$ is furthermore non-degenerate, then $\mathfrak{g}$ is reductive i.e.* $\mathrm{rad}(\mathfrak{g}) = Z(\mathfrak{g})$.

Proof (Sketch). $\mathrm{rad}(\mathfrak{g}) \supset Z(\mathfrak{g})$ is always true, so all that remains is to show the reverse containment. Show $[\mathfrak{g}, \mathrm{rad}(\mathfrak{g})] = 0$.
(1) $x \in [\mathfrak{g}, \mathrm{rad}(\mathfrak{g})]$ acts by 0 on any irreducible W (the proof of this is omitted. This implies that $x \in \ker(b_W)$.
(2) For

$$0 \to W' \to V \to W \to 0$$

we have that $b_V = b_{W'} + b_W$ by

$$\mathrm{Tr}\left(\begin{pmatrix} X_{W'} & * \\ 0 & X_W \end{pmatrix}\begin{pmatrix} Y_{W'} & * \\ 0 & Y_W \end{pmatrix}\right) = \mathrm{Tr}(X_{W'}Y_{W'}) + \mathrm{Tr}(X_W Y_W)$$

(3) By induction on $\dim V$ we show that $x \in \ker(b_V) = \{0\} \Rightarrow x = 0$. ▫

Theorem 27.6. *Each classical Lie algebra $\mathfrak{g} \subset \mathfrak{gl}(n, \mathbb{K})$ is reductive.*

Proof. $b_{\mathbb{K}^n}$ is non-degenerate $\Rightarrow \mathfrak{g}$ reductive. ▫

Cartan Criteria

Theorem 27.7 (Cartan's Criteria)**.**

a) $\mathfrak{g}$ *solvable iff* $\kappa(\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]) = 0$*;*

b) $\mathfrak{g}$ *semi-simple iff* κ *non-degenerate.*

Proof.
a) This proof omitted.
b) κ is non-degenerate $\Rightarrow \mathfrak{g}$ is reductive, i.e $\mathrm{rad}\,(\mathfrak{g}) = Z(\mathfrak{g})$. So now we show that $Z(\mathfrak{g}) = 0$ i.e.

$$\mathfrak{g} = \mathfrak{g}_{\mathrm{ss}} \oplus \mathrm{rad}\,(\mathfrak{g}) = \mathfrak{g}_{\mathrm{ss}} \oplus Z(\mathfrak{g}) = \mathfrak{g}_{\mathrm{ss}}$$

Let $x \in Z(\mathfrak{g}) \Rightarrow \mathrm{ad}(x) = 0 \in End(\mathfrak{g})$. $\mathrm{Tr}(\mathrm{ad}(x)\,\mathrm{ad}(y)) = 0$ for every $y \in \mathfrak{g} \Rightarrow \kappa(x, y) = 0$ $\forall y \in \mathfrak{g} \Rightarrow x = 0$.

For the reverse implication, $\mathfrak{g} = \mathfrak{g}_{\mathrm{ss}} \supset I = \ker(\kappa)$. Claim: I is solvable.

Proof of claim. Verify $\kappa|_{I\times I} = \kappa^I$ is the Killing form on I. (exercise) $\kappa^I([I, I], I) = 0$. By a) I is solvable. ▫

By $\mathfrak{g}$ semi-simple, $0 = I = \ker(\kappa)$. Thus, κ is non-degenerate. ▫

Example 27.8. $\kappa_{\mathfrak{sl}(2,\mathbb{C})}$ is represented by

$$\begin{pmatrix} 0 & 4 & 0 \\ 4 & 0 & 0 \\ 0 & 0 & 8 \end{pmatrix} \Rightarrow \ \mathfrak{sl}(2, \mathbb{C}) \text{ semi-simple}$$

28 March 21st, 2018

(Abstract)[‡‡]Jordan decomposition

[‡‡]By Abstract we mean basis independent

Recall $X \in M_n(\mathbb{C})$ has a Jordon canonical form if

$$x \sim \begin{bmatrix} \lambda_1 & 1 & & & & & & & \\ & \ddots & \ddots & & & & & & \\ & & \ddots & 1 & & & & & \\ & & & \lambda_1 & & & & & \\ & & & & \ddots & & & & \\ & & & & & \lambda_k & 1 & & \\ & & & & & & \ddots & \ddots & \\ & & & & & & & \ddots & 1 \\ & & & & & & & & \lambda_k \end{bmatrix}.$$

Note.

$$x \sim \begin{bmatrix} \lambda_1 & & & & & & 0 \\ & \ddots & & & & & \\ & & \lambda_1 & & & & \\ & & & \ddots & & & \\ & & & & \lambda_k & & \\ & & & & & \ddots & \\ 0 & & & & & & \lambda_k \end{bmatrix} + \begin{bmatrix} 0 & 1 & & & & & 0 \\ & \ddots & \ddots & & & & \\ & & \ddots & 1 & & & \\ & & & \ddots & & & \\ & & & & \ddots & 1 & \\ & & & & & \ddots & 1 \\ 0 & & & & & & 0 \end{bmatrix}$$

where the first is diagonal and the second is nilpotent, and these matrices commute.

Definition. $A \in End(V)$, V vector space $/\mathbb{C}$ is *semi-simple* if A is diagonalizable.

Theorem 28.1 (Jordan Decomposition)**.** *V finite dimensional vector space over $\mathbb{C}$. $A \in End(V)$. Then*

a) *There exists unique linear maps $A_s, A_n \in End(V)$ such that*

   i) $A = A_s + A_n$

   ii) $A_s A_n = A_n A_s$

   iii) *A_s is semi-simple and A_n is nilpotent.*

b) *There exists $P(T) = P_A(T) \in \mathbb{C}[T]$ and $Q(T) = Q_A(T) \in \mathbb{C}[T]$ such that*

   i) *$A_s = P(A)$ and $A_n = Q(A)$*

   ii) *$\gcd(P(T), Q(T)) = T$ in particular $P(0) = Q(0) = 0$. Hence, if $B \in End(V)$ commutes with A, then B commutes with A_s and A_n.*

c) $P_{A*}(T) = \overline{P_A(T)}$ *and similarly for* Q*. Where* A^* *is the hermitian adjoint with respect to some non-degenerate hermitian form:*

$$(Av, w) = (v, A^* w)$$

Also

$$\begin{aligned}(A^*)_s &= (A_s)^* \\ (A^*)_n &= (A_n)^*\end{aligned}$$

d)

$$\begin{aligned}(\operatorname{ad} A)_S &= \operatorname{ad}(A_s) \\ (\operatorname{ad} A)_n &= \operatorname{ad}(A_n)\end{aligned}$$

Proof. a,b) $A \in End(V)$ define a $\mathbb{C}$-algebra morphism

$$\begin{aligned} ev_A \colon \mathbb{C}[T] &\to End(V) \\ \sum c_n T^n &\mapsto \sum C_n A^n \end{aligned}$$

$\Rightarrow\ c_A(T) = \prod (T - a_i)^{m_i}$ wher $a_i \in \mathbb{C}$. The Chinese Remainder theorem gives:

$$\frac{\mathbb{C}[T]}{(c_A(T))} \cong \bigoplus_i \frac{\mathbb{C}[T]}{((T - a_i)^{m_i}) = I_i}$$

by

$$f(T) + (c_A(T)) \mapsto (f(T) + I_1, \ldots, f(T) + I_k).$$

So the system $\begin{cases} P(T) \equiv a_i \mod I_i & \forall i \\ P(T) \equiv 0 \mod T \end{cases}$ has to be a solution $\mod (c_A(T))$. Put $Q(T) = T - P(T)$ and $A_s \colon P(A),\ A_n = Q(A)$. correspondingly,

$$V = \bigoplus_{i=1}^{k} V_i$$

where $V_i = \{v \in V \mid (A - a_i)^{m_i} v = 0\}$. Then $A_s|_{V_i} = a_i \operatorname{Id}_{V_i}\ \forall i$ i.e. $(A - a_i \operatorname{Id}_{V_i} = 0$ or V_i. This implies A_s is diagonal one each $V_i \Rightarrow A_s$ is semi-simple.

Then $A_n = A - A_s$ is nilpotent on each V_i

$$(A_n|_{V_i})^{m_i} = (A|_{V_i} - A_s|_{V_i})^{m_i} = (A - a_i)^{m_i} = 0.$$

Hence $A = A_s + A_n$ by construction $[A_s, A_n] = [P(A), Q(A)] = 0$. To show uniqueness, suppose $A = S + N$ also. Then $A = A_s + A_n = S + N \Rightarrow A_s - S = N - A_n$ is both semi-simple and nilpotent $\Rightarrow$ they are 0.

c) $A^* = (A_s + A_n)^* = (A_s)^* + (A_n)^*$

Note. $(A_s)^*$ semi-simple, $(A_n)^*$ nilpotent, and they commute. So by uniqueness, $A_s^* = (A_s)^*$ and $A_n^* = (A_n)^*$.

d) $\operatorname{ad} A = \operatorname{ad}(A_s + A_n) = \operatorname{ad}(A_s) + \operatorname{ad}(A_n)$ as in part c) we get $\operatorname{ad}(A_s)$ is semi-simple, $\operatorname{ad}(A_n)$ nilpotent, and they commute. So by uniqueness $(\operatorname{ad} A)_s = \operatorname{ad}(A_s)$ and $(\operatorname{ad} A)_n = \operatorname{ad}(A_n)$. □

If A is semi-simple (nilpotent), then so is $\operatorname{ad} A$ (a Lemma in the text).

Note. If $\psi\colon End(V) \to End(W)$ is linear and satisfies A is semi-simple (nilpotent) implies $\psi(A)$ is semi-simple (nilpotent), and $[A, B] = 0 \Rightarrow [\psi(A), \psi(B)] = 0$ then:

$$\begin{aligned}\psi(A)_s &= \psi(A_s),\\ \psi(A)_n &= \psi(A_n).\end{aligned}$$

29 March 23rd, 2018

Homework 4 discussion

Problem 3. Let λ and μ be non-negative integers, $\lambda \geqslant \mu$. Prove the following isomorphism of $\mathfrak{sl}(2, \mathbb{C})$-representations:

$$V_\lambda \otimes V_\mu \cong V_{\lambda+\mu} \oplus V_{\lambda+\mu-2} \oplus V_{\lambda+\mu-4} \oplus \cdots \oplus V_{\lambda-\mu}$$

Hint: Find an injective morphism from one side to the other. Then check that the dimensions agree. (Start with $\mu = 1$.)

Solution. First we show that the dimensions agree that is

$$(*) \qquad (\lambda+1)(\mu+1) = \sum_{j=0}^{\min\{\lambda,\mu\}} \lambda + \mu - 2j + 1.$$

A combinatorial proof of $(*)$ is to double count a lattice of $\lambda + 1$ by $\mu + 1$ dots. So it suffices to find an injective intertwining operator

$$\Phi_j\colon V_{\lambda+\mu-2j} \to V_\lambda \otimes V_\mu \quad \forall j.$$

Let $v_{\lambda+\mu-2j}$ be the highest weight vector of $V_{\lambda+\mu-2j}$:

$$\begin{aligned}h.v_{\lambda+\mu-2j} &= (\lambda+\mu-2j)v_{\lambda+\mu-2j}\\ e.v_{\lambda+\mu-2j} &= 0\end{aligned}$$

We now show that $\Phi_j(v_{\lambda+\mu-2j} \in V_\lambda \otimes V_\mu$ satisfies these same identities.

We have a basis for $V_\lambda \otimes V_\mu$ is $\{v_\lambda^k \times v_\mu^\ell\}_{\ell=0,\dots,\mu}^{k=0,\dots,\lambda}$ where $v_\lambda^k = f^k v_\lambda$ (similarly for v_μ^ℓ).

$$h.(v_\ell^k \otimes v_\mu^\ell) = h.v_\lambda^k \otimes v_\mu^\ell + v_\lambda^k \otimes h.v_\mu^\ell = ((\lambda-2k) + (\mu-2\ell))v_\lambda^k \otimes v_\mu^\ell,$$

so $k + \ell = j$. Hence,

$$\Phi_j(v_{\lambda+\mu-2j}) = \sum_{k+\ell=j} \xi_{k,\ell} v_\lambda^k \otimes v\mu^\ell.$$

Now we see what the $\xi_{k,\ell}$ are.

$$\begin{aligned} 0 &= e.\left(\sum_{k+\ell=j} \xi_{k,\ell} v_\lambda^k \otimes v_\mu^\ell\right) \\ &= \sum_{k+\ell=j} \xi_{k,\ell}\Big((\lambda+1-k)v_\lambda^{k-1} \otimes v_\mu^\ell + v_\lambda^k \otimes (\mu+1-\ell)v_\mu^{\ell-1}\Big) \\ &= \sum_{\ell+k=j-1} \Big(\xi_{k+1,\ell}(\lambda-k) + \xi_{k,\ell+1}(\mu-\ell)\Big) v_\lambda^k \otimes v_\mu^\ell \end{aligned}$$

This implies that $\xi_{k+1,\ell}(\lambda - k) + \xi_{k,\ell+1}(\mu - \ell) = 0$ for all $k + \ell = j - 1$. With some more work we can see that Φ_j is an injective intertwining operator. The formula $V_\lambda \otimes V_\mu \cong \oplus \cdots$ is the Clebsch-Gordon formula.

Problem 5. Let $\mathfrak{g}$ be Lie algebra over $\mathbb{C}$ and let $(\cdot, \cdot)$ be a non-degenerate, symmetric, $\mathfrak{g}$-invariant, bilinear form on $\mathfrak{g}$. Let $\mathfrak{g}[t, t^{-1}]$ be the vector space of all Laurent polynomial expressions in t with coefficients from $\mathfrak{g}$. Define

$$\tilde{\mathfrak{g}} = \mathfrak{g}[t, t^{-1}] \oplus \mathbb{C}c$$

where c is a new element, and define a $\mathbb{C}$-bilinear bracket on $\tilde{\mathfrak{g}}$ by requiring

$$[xt^m, yt^n] = [x, y]t^{m+n} + \delta_{m+n,0} \cdot m \cdot (x, y) \cdot c, \qquad x, y \in \mathfrak{g},\ m, n \in \mathbb{Z}$$

$$[X, c] = 0 = [c, X], \qquad \forall X \in \tilde{\mathfrak{g}}$$

(a) Prove that this makes $\tilde{\mathfrak{g}}$ into a Lie algebra.

(b) Prove that if $\mathfrak{g}$ is nilpotent, then so is $\tilde{\mathfrak{g}}$.

$\tilde{\mathfrak{g}} = \mathfrak{g}[t, t^{-1}] \oplus \mathbb{C}c$ (Kac Moody Lie algebra) is called the affine Lie algebra associated to $\mathfrak{g}$. $\tilde{\mathfrak{g}}/\mathbb{C}c \cong \mathfrak{g}[t, t^{-1}]$ with bracket

$$[xt^m, yt^n] = [x, y]t^{m+n}$$

is the Loop algebra of $\mathfrak{g}$, and the loop algebra is the Lie algebra of the loop group $LG = \{\gamma \colon S^1 \to G \text{ is a morphism of manifolds}\}$. The Loop group is an "$\infty$-dimensional Lie group", and plays an important role in physics.

Problem 4. If $x \in \mathfrak{g}$, extend $\operatorname{ad} x$ to an endomorphism of $U(\mathfrak{g})$ by defining $\operatorname{ad} x(y) = xy - yx$ for $y \in U(\mathfrak{g})$. Prove that if $\dim \mathfrak{g} < \infty$, the each element of $U(\mathfrak{g})$ belongs to a finite-dimensional $\mathfrak{g}$-submodule of $U(\mathfrak{g})$.

Definition. A *derivation* $D \colon A \to A$ of an algebra A is a linear map such that

$$D(ab) = D(a)b + aD(b).$$

Definition. A derivation D is *locally nilpotent* if $\forall a \in A\ \exists N > 0$ such that $D^N(a) = 0$.

Example 29.1. $D = \dfrac{d}{dx}$ on $A = \mathbb{C}[x]$.

Definition. A linear map $L\colon V \to V$ is *locally finite* if $\forall v \in V$, $\dim \operatorname{span}\{L^n(v) \mid n \in \mathbb{N}\} < \infty$.

The idea of how to find the solution is consider the following. By the PBW theorem (Theorem 23.1) we have,

$$u = \sum_{k \in \mathbb{N}^n} c_k x_1^{k_1} \cdots x_n^{k_n} \in U_{(N)}(\mathfrak{g})$$

where $n = \dim \mathfrak{g}$ and $N = \max\{\sum k_i\}$. The key realization is that

$$(\operatorname{ad} x)(u) = xu - ux \equiv 0$$

modulo lower degree terms with respect to the PBW filtration, i.e. $xu - ux \in U_{(n-1)}$.

30 March 26th, 2018

Semi-simple Lie Algebras

Recall:

- $\mathfrak{g}$ is a *simple* Lie algebra if the only proper ideal is 0 (and $\mathfrak{g}$ is non-abelian).
- $\mathfrak{g}$ is *semi-simple* if the only proper solvable ideal is 0 (and $\mathfrak{g}$ is non-abelian).

Lemma 30.1. *If $\mathfrak{g}$ is a real Lie algebra, then $\mathfrak{g}$ is semi-simple iff its complexification (i.e. $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \oplus i\,\mathfrak{g} \cong \mathbb{C} \otimes_{\mathbb{R}} \mathfrak{g}$) is.*

Proof. We use Cartan's Criterion (Theorem 27.7):

Let κ be the killing form on $\mathfrak{g}$.

$$\kappa(x, y) = \operatorname{Tr}_{\mathfrak{g}}(\operatorname{ad} x \operatorname{ad} y) \qquad \forall x, y \in \mathfrak{g}$$

We can extend κ to a $\mathbb{C}$-bilinear form $\kappa_{\mathbb{C}}$ on $\mathfrak{g}_{\mathbb{C}}$:

$$\begin{aligned}\kappa_{\mathbb{C}}(x + iy, z + iw) &= \kappa_{\mathbb{C}}(x, z) + i\kappa_{\mathbb{C}}(x, w) + i\kappa_{\mathbb{C}}(y, z) + i^2\kappa_{\mathbb{C}}(y, w)\\ &:= \kappa(x, z) + i\kappa(x, w) + \kappa(y, z) + i^2\kappa(y, w).\end{aligned}$$

However, $\mathfrak{g}_{\mathbb{C}}$ has its own killing form $\kappa_{\mathfrak{g}_{\mathbb{C}}}$. We claim that $\kappa_{\mathfrak{g}_{\mathbb{C}}} = \kappa_{\mathbb{C}}$.

Proof of Claim.

$$\begin{aligned}\kappa_{\mathfrak{g}_{\mathbb{C}}}(x + iy, z + iw) &= \operatorname{Tr}_{\mathfrak{g}_{\mathbb{C}}}(\operatorname{ad}(x + iy) \operatorname{ad}(z + iw))\\ &= \operatorname{Tr}_{\mathfrak{g}_{\mathbb{C}}}(\operatorname{ad} x \operatorname{ad} z) + i \operatorname{Tr}_{\mathfrak{g}_{\mathbb{C}}}(\operatorname{ad} x \operatorname{ad} w) + \cdots\\ &\overset{*}{=} \kappa(x, z) + i\kappa(x, w) + \cdots\end{aligned}$$

Where $*$ is due to $\mathrm{Tr}_{\mathfrak{g}_{\mathbb{C}}}(\operatorname{ad} x \operatorname{ad} y) = \mathrm{Tr}_{\mathfrak{g}}(\operatorname{ad} x \operatorname{ad} y)\ \forall x, y \in \mathfrak{g}$. □

So

$$\begin{aligned} \mathfrak{g} \text{ is semi-simple} &\Leftrightarrow \kappa \text{ is non-degenerate} \\ &\Leftrightarrow \kappa_{\mathbb{C}} \text{ is non-degenerate} \\ &\Leftrightarrow \kappa_{\mathfrak{g}_{\mathbb{C}}} \text{ is non-degenerate} \\ &\Leftrightarrow \mathfrak{g}_{\mathbb{C}} \text{ is semi-simple} \end{aligned}$$

□

Proposition 30.2. *If $\mathfrak{g}$ is semi-simple, and $I \subseteq \mathfrak{g}$ is an ideal, then $I^{\perp} = \{x \in \mathfrak{g} \mid \kappa(x,y) = 0\ \forall y \in I\}$ is an ideal of $\mathfrak{g}$, and $\mathfrak{g} = I \oplus I^{\perp}$.*

Proof. $I^{\perp}$ is an ideal by Lemma 27.2. We claim that $I \cap I^{\perp} = \{0\}$. Indeed $I \cap I^{\perp}$ is an ideal such that $\kappa_{I \cap I^{\perp}} = 0$ because $\kappa_{I \cap I^{\perp}} = \kappa|_{(I \cap I^{\perp}) \times (I \cap I^{\perp})}$ by Exercise 5.1 in the text. So by Cartan's criteria for solvability (Theorem 27.7), $I \cap I^{\perp}$ is solvable. Hence $I \cap I^{\perp} = 0$ since $\mathfrak{g}$ is semi-simple. $\dim I \oplus I^{\perp} = k + (n-k) = n$ if $\dim \mathfrak{g} = n$, $\dim I = k$. Thus $\mathfrak{g} = I \oplus I^{\perp}$. □

Corollary 30.3. *Let $\mathfrak{g}$ be a (finite dimensional) Lie algebra. Then $\mathfrak{g}$ is semi-simple iff $\mathfrak{g} = I_1 \oplus I_2 \oplus \cdots I_k$, where I_j are simple ideals.*

Proof. Suppose $\mathfrak{g}$ is semi-simple. If $\mathfrak{g}$ is simple we are done: $k = 1$, and $I_1 = \mathfrak{g}$. Otherwise Let $I \subseteq \mathfrak{g}$ a proper nonzero ideal. By Proposition 30.2 $\mathfrak{g} = I \oplus I^{\perp}$. By Exercise 5.1 in the text, the restriction to $I \times I$ (and $I^{\perp} \times I^{\perp}$) is non-degenerate, so I (respectively $I^{\perp}$) is semi-simple. It now follows by induction on the $\dim \mathfrak{g}$. □

Corollary 30.4. $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$ *if $\mathfrak{g}$ is semi-simple.*

Proof.

$$\begin{aligned} [\mathfrak{g}, \mathfrak{g}] &= [\oplus_j I_j, \oplus_k I_k] \\ &= \oplus_{j,k} [I_j, I_k] \qquad \text{for } j \neq k\ [I_j, I_k] \subseteq I_j \cap I_k = 0 \\ &= \oplus_j [I_j, I_j] \\ &= \oplus I_i \qquad = \mathfrak{g}\,. \end{aligned}$$

□

Corollary 30.5. *If $\mathfrak{g}$ is semi-simple, then $\mathfrak{g}/I$ is semi-simple for any ideal I.*

Proof. $\mathfrak{g} = \oplus_j \mathfrak{g}_j$ $\mathfrak{g}_j$ simple ideals. By a proposition in the book, any ideal $I \subseteq \mathfrak{g}$ has the form $I = \oplus_j I_j$ where I_j ideal of $\mathfrak{g}_j$. So $I_j = 0$ or $\mathfrak{g}_j$. Hence, $\mathfrak{g}_I \oplus_{\{j \mid I_j \neq 0\}} \mathfrak{g}_j$. □

Example 30.6. The only ideals of $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{C}) \times \mathfrak{sl}(3,\mathbb{C})$ are 0×0, $0 \times \mathfrak{sl}(3,\mathbb{C})$, $\mathfrak{sl}(2,\mathbb{C})$, and $\mathfrak{g}$.

Example 30.7. So $\dfrac{\mathfrak{g}}{\mathfrak{sl}(2,\mathbb{C}) \times 0} \cong \mathfrak{sl}(3,\mathbb{C})$.

Kirillov 6.2 Study on your own

Complete reducibility of representations of semi-simple Lie algebras (Kirillov 6.3)

The Casimir Element

Let $\mathfrak{g}$ be a Lie algebra, $B\colon \mathfrak{g} \times \mathfrak{g} \to \mathbb{K}$ a non-degenerate, symmetric, invariant bilinear form. Let $\{x_i\}$ be any basis of $\mathfrak{g}$. Let $\{x^i\}$ be the dual basis of $\mathfrak{g}$ with respect to B (i.e. $B(x_i, x^j) = \delta_{ij}\ \forall i, j$).

Definition. The *casimir element* c_B is given by

$$c_B = \sum_{i=1}^{n} x_i x^i.$$

Theorem 30.8. *c_B is independent of the choice of basis, and belongs to the center of $U(\mathfrak{g})$.*

31 March 28th, 2018

Weyl's Theorem

Let $\mathfrak{g}$ be a finite dimensional complex semi-simple Lie algebra and V a finite dimensional complex representation of $\mathfrak{g}$

Theorem 31.1 (Weyl's Theorem)**.** *For $\mathfrak{g}$ and V as above, V is completely reducible:*

$$V \cong \bigoplus_{i=1}^{n} V_i$$

where V_i are irreducible representations.

To prove this we need the following two lemmas.

Lemma 31.2. *Let C_V be the casimir element for $\overline{\rho}\colon \dfrac{\mathfrak{g}}{\ker\rho} \to \mathfrak{gl}(V)$ i.e. Let $\mathfrak{g}' = \mathfrak{g} / \ker\rho$ (then $\mathfrak{g}'$ is semi-simple). Pick a basis $\{x_i\}$ for $\mathfrak{g}'$ and a dual basis $\{x^i\}$ with respect to $B(x, y) = \operatorname{Tr}(\overline{\rho}(x)\overline{\rho}(y))$, $c_V = \sum x_i x^i \in U(\mathfrak{g}')$. Then:*

(i) $\operatorname{Tr}(c_V|_V) = \dim \mathfrak{g} / \ker\rho \neq 0$ *iff ρ is nontrivial.*

(ii) *If V is irreducible, then $c_V = \lambda \operatorname{Id}_V$, and $\lambda = \dfrac{\dim(\mathfrak{g} / \ker\rho)}{\dim V}$.*

Proof. (i) Consider the following:

$$\begin{aligned}
\operatorname{Tr}(c_V|_V) &= \operatorname{Tr}(\overline{\rho}(c_V)) \\
&= \operatorname{Tr}\left(\sum \overline{\rho}(x_i)\overline{\rho}(x^i)\right) \\
&= \sum_i B_V(x_i, x^i) \\
&= \sum_i 1 \\
&= \dim \mathfrak{g}' .
\end{aligned}$$

(ii) Claim: $c_V \in Z(U(\mathfrak{g}'))$ i.e. $[C_v, x] = 0 \ \forall x \in \mathfrak{g}'$ i.e. $c_V \in U(\mathfrak{g}')^{\mathfrak{g}'}$[§§].

We have:

$$End(\mathfrak{g}') \xrightarrow{\cong} \mathfrak{g}' \otimes \mathfrak{g}'^* \xrightarrow{\cong} \mathfrak{g}' \otimes \mathfrak{g}' \to U(\mathfrak{g}')$$

We can show that $\mathrm{Id}_{\mathfrak{g}'} \mapsto c_V$ (exercise). Moreover, using B_V is $\mathfrak{g}'$-invariant the map $End(\mathfrak{g}') \to U(\mathfrak{g}')$ is an intertwining operator. In particular, $End(\mathfrak{g}')^{\mathfrak{g}'} \to U(\mathfrak{g}')^{\mathfrak{g}'}$, so $\mathrm{Id}_{\mathfrak{g}'} \in End(\mathfrak{g}')^{\mathfrak{g}'} \Rightarrow c_V \in U(\mathfrak{g}')^{\mathfrak{g}'}$. By Schur's Lemma (Lemma 18.1), $c_V|_V = \lambda \, \mathrm{Id}_V$ for some $\lambda \in \mathbb{C}$. Taking the trace and using (i), we are done. ▫

Lemma 31.3. *If* $\dim V = 1$*, then* ρ *is the trivial representation:* $\rho(x) = 0$ *for all* $x \in \mathfrak{g}$*.*

Proof. $\rho(\mathfrak{g}) = \rho([\mathfrak{g}, \mathfrak{g}]) = [\rho(\mathfrak{g}), \rho(\mathfrak{g})] = 0$. Since $\rho(\mathfrak{g}) \subseteq \mathfrak{gl}(V) \cong \mathbb{C}$. ▫

Proof of Weyl's Theorem (31.1). We want to show that for every subrepresentation $W \subseteq V$, there exists a subrepresentation W' such that $V = W \oplus W'$.

<u>Step 0</u>
If $\rho\colon \mathfrak{g} \to \mathfrak{gl}(V)$ is the trivial representation, then any linear complement W' of W will be a subrepresentation of V.

We proceed by induction on $\dim V$.

<u>Step 1</u>
Suppose W has codim 1. Then V/W has dim 1, so by lemma 31.3, $\mathfrak{g}$ acts trivially on V/W.
<u>Step 1 a)</u> Suppose W has a proper nonzero subrepresentation W'. Then

$$\mathbb{C} \equiv V/W \cong \frac{V/W'}{W/W'}.$$

By induction, W/W' has an invariant complement i.e. there exists suprepresentation $\tilde{W}/W' \subseteq V/W'$ such that $V/W' = W/W' \oplus \tilde{W}/W'$. $\dim \tilde{W} = 1 \dim W' \leqslant \dim W < \dim V$ So by induction, $W = W' \oplus X$, for some 1 dimensional subrepresentation $X \subset \tilde{W}$. Then $V \cong W \oplus X$ since the dimensions add up and

$$W \cap X \subseteq (W \cap \tilde{W}) \cap X \subseteq W' \cap X = 0.$$

[§§] $\mathfrak{g}'$ acts on $U(\mathfrak{g}')$ by ad where $x.w = (\operatorname{ad} x)(w)$.

Step 1 b) Suppose that W is irreducible. Let $c = c_V$ be the casimir element. Then $c(W) \subseteq W$, since W is a subrepresentation of V and $\ker \rho(c)$ is a subrepresentation of V since $[\rho(c), \rho(x)] = 0$ for all $x \in \mathfrak{g}$. Since $\mathfrak{g}$ acts trivially on V/W. $c(V/W) = V$.

On the other hand $c|_W = \lambda \operatorname{Id}_W$ for some $\lambda \in \mathbb{C}$ (by Schur's Lemma [Lemma 18.1] or Lemma 31.3). $0 \neq \operatorname{Tr} c = 0 + \lambda$, $\ker c$ is a 1 dimensional subrepresentation of V, $(\ker c) \cap W = 0$. Hence $V = W \oplus (\ker c)$.

Step 2
$0 \neq W \subseteq V$ arbitrary subrepresentation. Consider

$$\mathscr{V} = \{T \in \operatorname{Hom}(V, W) \mid T|_W = \lambda \operatorname{Id}_W \text{ some } \lambda \in \mathbb{C}\}$$

and

$$\mathscr{W} = \{T \in \operatorname{Hom}(V, W) \mid T|_W = 0\} \subseteq \mathscr{V}$$

Claim: $\mathscr{V}$ and $\mathscr{W}$ are subrepresentations of $\operatorname{Hom}(V, W)$. If $T \in \mathscr{V}$, $x \in \mathfrak{g}$, $w \in W$, then

$$(x.T)(w) = (x.(T(w)) - T(x.w) = \lambda x.w - \lambda x.w = 0.$$

So $x.\mathscr{V} \subseteq \mathscr{W}$.

Note. $\mathscr{V}/\mathscr{W}$ has $\dim 1$.

By Step 1 $\mathscr{V} = \mathscr{W} \oplus \mathbb{C}T$

$$T \colon V \to W, \qquad T|_W = \lambda \operatorname{Id}_W$$

$T \rightsquigarrow \frac{1}{\lambda} T$, without loss of generality $\lambda = 1$. So $T|_W = \operatorname{Id}_W \Rightarrow T^2 = T$. $x.T = 0$ for all x since $\mathbb{C}T$ is a subrepresentation of $\mathscr{V}$.

$$\begin{aligned} &\Rightarrow (x.T)(v) = 0 \qquad \forall v \in V \\ &\Rightarrow x.(T(v)) - T(x.V) = 0 \\ &\Rightarrow T \in \operatorname{Hom}_{\mathfrak{g}}(V, W) \text{ is an intertwining operator.} \end{aligned}$$

So $\ker T$ is a subrepresentation of V. $V = W \oplus \ker T$, $V = T(V) + (1 - T)(V)$. Where $T \in W$ and $(1 - T) \in \ker T$. $\square$

32 March 30th, 2018

Toral Subalgebras

Let $\mathfrak{g}$ finite dimensional semi-simple Lie algebra over $\mathbb{C}$.

Proposition 32.1. *Any derivation of $\mathfrak{g}$ is inner i.e. if D is a derivation, then $D = \operatorname{ad} x$, for some $x \in \mathfrak{g}$.*

Proof. $\operatorname{ad}\mathfrak{g}$ is an ideal of $\operatorname{Der}(\mathfrak{g})$, and $\operatorname{Out}\mathfrak{g} := \dfrac{\operatorname{Der}(\mathfrak{g})}{\operatorname{ad}(\mathfrak{g})}$. Also $\operatorname{ad}\colon \mathfrak{g} \to \operatorname{ad}\mathfrak{g}$ is injective, since $\ker(\operatorname{ad}) = \mathcal{Z}(\mathfrak{g}) = 0$ by $\mathfrak{g}$ semi-simple. Let $I = (\operatorname{ad}\mathfrak{g})^{\perp}$ with respect to the killing form κ on $\operatorname{Der}\mathfrak{g}$.Now $\kappa|_{(\operatorname{ad}\mathfrak{g}\times\operatorname{ad}\mathfrak{g})}$ coincides with the killing form on $\operatorname{ad}\mathfrak{g}$ (by Exercise 5.1 in Kirillov). Now

$$\begin{aligned}
\mathfrak{g} \text{ semi-simple} &\Rightarrow \operatorname{ad}\mathfrak{g} \text{ semi-simple}\\
&\Rightarrow \kappa|_{\operatorname{ad}\mathfrak{g}\times\operatorname{ad}\mathfrak{g}} \text{ is non-degenerate}\\
&\Rightarrow I \cap (\operatorname{ad}\mathfrak{g}) = 0\\
&\Rightarrow [I, \operatorname{ad}\mathfrak{g}] \subseteq I \cap (\operatorname{ad}\mathfrak{g}) = 0\\
&\Rightarrow \text{If } \delta \in I \quad \operatorname{ad}(\delta(x)) = [\delta, \operatorname{ad}x] = 0 \text{ for all } x \in \mathfrak{g}\\
&\Rightarrow \delta(x) = 0\ \forall x \in \mathfrak{g}\\
&\Rightarrow I = 0\\
&\Rightarrow \operatorname{ad} \text{ is injective.}
\end{aligned}$$

Thus $\operatorname{Der}\mathfrak{g} = \operatorname{ad}\mathfrak{g}$. □

Definition. $x \in \mathfrak{g}$ is *nilpotent* (respectively *semi-simple*) if $\operatorname{ad}x$ is nilpotent (respectively semi-simple).

Theorem 32.2 (Jordon decomposition in $\mathfrak{g}$)**.** *For all $x \in \mathfrak{g}$, there exists $x_n,\ x_s \in \mathfrak{g}$ such that*

$$\begin{aligned}
\operatorname{ad}(x_n) &= (\operatorname{ad}x)_n\\
\operatorname{ad}(x_s) &= (\operatorname{ad}x)_s
\end{aligned}$$

Thus, $x = x_s + x_n$, $[x_s, x_n] = 0$, and if $y \in \mathfrak{g}$ and $[x, y] = 0$, then $[x_n, y] = [x_s, y] = 0$.

Proof. Fix $x \in \mathfrak{g}$. $\mathfrak{g} = \bigoplus_{\lambda\in\mathbb{C}} \mathfrak{g}_\lambda$ where $\mathfrak{g}_\lambda = \{y \in \mathfrak{g} \mid \exists N > 0 : (\operatorname{ad}x - \lambda\operatorname{Id})^N(y) = 0\}$. We need the following lemma from the text:

Lemma 32.3. $[\mathfrak{g}_\lambda, \mathfrak{g}_\mu] \subset \mathfrak{g}_{\lambda+\mu}$.

We have that $(\operatorname{ad}x)_s|_{\mathfrak{g}_\lambda} = \lambda \cdot \operatorname{Id}_{\mathfrak{g}_\lambda}$. The lemma implies that $(\operatorname{ad}x)_s\colon \mathfrak{g} \to \mathfrak{g}$ is a derivation. By proposition, $(\operatorname{ad}x)_s = \operatorname{ad}x_s$ for some $x_s \in \mathfrak{g}$. Then $x_n = x - x_s \;\Rightarrow\; (\operatorname{ad}x)_n = \operatorname{ad}(x_n)$.
□

Corollary 32.4. $\mathfrak{g} \neq 0$ *contains at least one semi-simple element* $\neq 0$.

Proof. If $x \in \mathfrak{g}$ is nilpotent for all $x \in \mathfrak{g}\backslash\{0\}$. This implies $\mathfrak{g}$ is nilpotent by Engel's theorem (theorem 25.3) Which implies $\mathfrak{g}$ is solvable which is a contradiction. □

Definition. A subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is *toral* if

i) $\mathfrak{h}$ is commutative (abelian): $[\mathfrak{h}, \mathfrak{h}] = 0$.

ii) every element of $\mathfrak{h}$ is semi-simple.

Theorem 32.5. *Let $\mathfrak{h} \subseteq \mathfrak{g}$ be a toral subalgebra. Then*

i) $\mathfrak{g} = \oplus_{\alpha \in \mathfrak{h}^*}$*, where* $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid [h, x] = \alpha(h)x \in \mathfrak{h}\}$ *and* $\mathfrak{h}^* = \mathrm{Hom}_{\mathbb{C}}(\mathfrak{h}, \mathbb{C})$.

ii) $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta} \qquad \forall \alpha, \beta \in \mathfrak{h}^*$

iii) $\alpha + \beta \neq 0 \;\Rightarrow\; \kappa(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0$.

iv) $\kappa_{\mathfrak{g}_\alpha \times \mathfrak{g}_{-\alpha}}$ *is non-degenerate.*

Proof. i) $\forall h \in \mathfrak{h}$, $\operatorname{ad} h$ is diagonalizable. Also $[\operatorname{ad} h_1, \operatorname{ad} h_1] = \operatorname{ad}[h_1, h_2] = 0$, so $\{\operatorname{ad} h \mid h \in \mathfrak{h}\}$ is simultaneously diagonalizable. Pick basis $\{h_1, \dots, h_r\}$ for $\mathfrak{h}$

$$\mathfrak{g} = \bigoplus_{\xi \in \mathbb{C}^r} \{x \in \mathfrak{g} \mid [h_i, x] = \xi_i x;\ i = 1, \dots, r\} =: \mathfrak{g}_\xi$$

$h = a_1 h_1 + \cdots + a_r h_r \;\Rightarrow\; [h, X] = (\sum a_i \xi_i)X$, hence $\xi \in \mathbb{C}^r \rightsquigarrow \alpha\,\mathfrak{h}^*$ by $\alpha(h_i) = \xi_i$ i.e. $\mathbb{C}^r \cong \mathfrak{h}^*$ as vector spaces. Then $\mathfrak{g}_\xi = \mathfrak{g}_\alpha$.

ii) For $x \in \mathfrak{g}_\alpha$, $y \in \mathfrak{g}_\beta$, and $h \in \mathfrak{h}$:

$$\begin{aligned}[h, [x, y]] &= [[h, x], y] + [x, [h, y]] \\ &= \alpha(h)[x, y] + \beta(h)[x, y] \\ &= (\alpha + \beta)(h)[x, y].\end{aligned}$$

iii) κ invariant shows us

$$0 = \kappa([h, x], y) + \kappa(x, [h, y]) = (\alpha + \beta)(h) \cdot \kappa(x, y).$$

iv) κ non-degenerate and iii) show that this is true. ▫

Lemma 32.6.

i) $\kappa|_{\mathfrak{g}_0 \times \mathfrak{g}_0}$ *is non-degenerate*

ii) $x \in \mathfrak{g}_0 \Rightarrow x_s, x_n \in \mathfrak{g}_0$

iii) $\mathfrak{g}_0$ *is* <u>*reductive in* $\mathfrak{g}$</u> *i.e.* $\operatorname{ad}\colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{h})$ *is completely reducible.*

33 April 2nd, 2018

Cartan Subalgebras

Definition. A *cartan subalgebra* is a toral subalgebra such that

$$C_{\mathfrak{g}}(\mathfrak{h}) = \{x \in \mathfrak{g} \mid [x, \mathfrak{h}] = 0\} = \mathfrak{h}\,.$$

Example 33.1. Let $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$, then $\mathfrak{h} = \{\text{diagonal matrices three } 0\}$ is a Cartan subalgebra. Indeed, $\mathfrak{h}$ is commutative, and if $h \in \mathfrak{h}$,

$$\begin{aligned} \operatorname{ad} h \colon \mathfrak{g} &\to \mathfrak{g} \\ x &\mapsto [h, x] \quad \forall x \in \mathfrak{g} \end{aligned}$$

is diagonalizable $h = \begin{bmatrix} h_1 & & 0 \\ & \ddots & \\ 0 & & h_n \end{bmatrix} \in \mathfrak{h} = \sum_i h_i E_{ii}$. Then

$$\begin{aligned} (\operatorname{ad} \mathfrak{h})(E_i j) &= [\mathfrak{h}, E_{ij}] \\ &= \sum h_k [E_{kk}, E_{ij}] \\ &= (h_i - h_j) E_{ij}. \end{aligned}$$

So $\mathfrak{h}$ is toral. Also, if $x \in C_{\mathfrak{g}}(\mathfrak{h})$, then $[h, x] = 0$ $\forall h \in \mathfrak{h}$. Pick h with distinct eigenvalues. This implies any eigenvector for h is an eigenvector for x. Which implies x is diagonal and thus in $x \in \mathfrak{h}$.

Theorem 33.2. *Any toral subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ which is maximal (with respect to inclusion) among all toral subalgebra is a Cartan subalgebra.*

Proof. Let $\mathfrak{h} \subset \mathfrak{g}$ be a maximal toral subalgebra.

$$\mathfrak{g} = \bigoplus_{\alpha \in \mathfrak{h}^*} \mathfrak{g}_\alpha, \qquad \mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid [h, x] = \alpha(h) x \ \forall h \in \mathfrak{h}\}.$$

Note, $C_{\mathfrak{g}}(\mathfrak{h}) = \mathfrak{g}_0$, and we claim that $\mathfrak{g}_0$ is toral. (Then, since $\mathfrak{h} \subset \mathfrak{g}_0$ and $\mathfrak{h}$ maximal $\Rightarrow$ $\mathfrak{h} = \mathfrak{g}_0$.) Let $x \in \mathfrak{g}_0$. Then $(\operatorname{ad} x)|_{\mathfrak{g}_0}$ is nilpotent. Otherwise, it has a nonzero eigenvalue and $(\operatorname{ad} x_s)|_{\mathfrak{g}_0} \neq 0$. This implies $x_s \notin \mathfrak{h}$, and $\mathfrak{h} \oplus \mathbb{C}_{x_s}$ is a toral subalgebra that strictly contains $\mathfrak{h}$ which is a contradiction. By Engel's Theorem (theorem 25.3), $\mathfrak{g}_0$ is nilpotent. By a lemma $\mathfrak{g}_0$ is reductive, and $\mathfrak{g}_0$ is commutative.

All that remains is to show that $\mathfrak{g}_0$ consists of semi-simple elements. Let $x \in \mathfrak{g}_0$. We want to show that $x_n = 0$. $\operatorname{ad} x_n$ is nilpotent and $\mathfrak{g}_0$ is commutative; therefore, $(\operatorname{ad} x_n)(\operatorname{ad} y)$ is nilpotent for every $y \in \mathfrak{g}_0$. Thus $\operatorname{Tr}\big((\operatorname{ad} x_n)(\operatorname{ad} y)\big) = 0 \qquad \forall y \in \mathfrak{g}_0$. Hence, $x_n = 0$ since $\kappa|_{\mathfrak{g}_0 \times \mathfrak{g}_0}$ is non-degenerate. □

Corollary 33.3. *Any (finite dimensional semi-simple Lie algebra over $\mathbb{C}$) $\mathfrak{g}$ contains at least on Cartan subalgebra.*

Fact: All are conjugate under $Aut(\mathfrak{g})$

Definition. The *rank* of a semi-simple Lie algebra $\mathfrak{g}$ is $\operatorname{rank} \mathfrak{g} = \dim \mathfrak{h}$ where $\mathfrak{h}$ is any Cartan subalgebra.

Example 33.4. $\operatorname{rank} \mathfrak{sl}(n, \mathbb{C}) = n - 1$

Root Space Decomposition

$\mathfrak{g}$ is finite dimensional semi-simple Lie algebra over $\mathbb{C}$ (e.g. $\mathfrak{sl}(n, \mathbb{C})$) and $\mathfrak{h} \subset \mathfrak{g}$ a fixed choice of Cartan subalgebra.

Definition. The decomposition

$$\mathfrak{g} = \bigoplus_{\alpha \in \mathfrak{h}^*} \mathfrak{g}_\alpha = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha$$

where $R = \{\alpha \in \mathfrak{h}^* \setminus \{0\} \mid \mathfrak{g}_\alpha \neq 0\}$ is called the *root space decomposition* of $\mathfrak{g}$. R is the *root system* of $\mathfrak{g}$, the elements are called *roots* of $\mathfrak{g}$, and $\mathfrak{g}_\alpha$ are the *root spaces*.

Theorem 33.5 (Theorem 6.39 in Kirillov)**.** *Let* $\mathfrak{g} = \oplus_{i=1}^n \mathfrak{g}_i$. *Then*

i) *Every Cartan subalgebra of* $\mathfrak{g}$ $\mathfrak{h} = \oplus_i \mathfrak{h}_i$ *where* $\mathfrak{h}_i$ *is a Cartan subalgebra of* $\mathfrak{g}_i$

ii) *Then* $R = \bigsqcup_i R_i$ *a disjoint union where* $R_i \subseteq (\mathfrak{h}_i)^* \hookrightarrow \oplus(\mathfrak{h}_i)^* = \mathfrak{h}^*$.

Example 33.6. $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$, $\mathfrak{h} = \text{diag} \cap \mathfrak{g}$, and $\varepsilon_i \in \mathfrak{h}^*$ by $\varepsilon_i(h) = \varepsilon_i\left(\begin{bmatrix} h_1 & & 0 \\ & \ddots & \\ 0 & & h_n \end{bmatrix}\right) = h_i$.

Then

$$\begin{aligned} h[h, E_{ij}] &= (h_i - h_j)E_{ij} \\ &= (\varepsilon_i(h) - \varepsilon_j(h))E_{ij} \\ &= (\varepsilon_i - \varepsilon_j)(h)E_{ij}. \end{aligned}$$

So $E_{ij} \in \mathfrak{g}_{\varepsilon_i - \varepsilon_j}$, $R = \{\varepsilon_i - \varepsilon_j \mid i \neq j\}$, and $\mathfrak{g}_{\varepsilon_i - \varepsilon_j} = \mathbb{C}E_{ij}$.

Example 33.7. If we consider specifically $\mathfrak{sl}(3, \mathbb{C})$ from the previous example, we see that there are 6 roots for $\mathfrak{sl}(3, \mathbb{C})$.

Goal: We want to bring in representation theory for $\mathfrak{sl}(2, \mathbb{C})$

Coroots

Definition. For each $\alpha \in R$, there is a corresponding *coroot* $\check{\alpha} = h_\alpha \in \mathfrak{h}$.

Let $(\ , \ )$ be an invariant non-degenerate symmetric bilinear form on $\mathfrak{g}$. We know that the restriction to $\mathfrak{h}$ is non-degenerate. This implies that $\mathfrak{h} \cong \mathfrak{h}^*$ by $h \mapsto (-, h)$. Let H_α be the inverse image of α under this map. Then

$$(H_\alpha, h) = \alpha(h) \qquad \forall h \in \mathfrak{h}\,.$$

Also convenient to define $\mathfrak{h}^*$ by $(\alpha, \beta) := (H_\alpha, H_\beta) = \alpha(H_\beta)$.

Lemma 33.8. $(\alpha, \alpha) = (H_\alpha, H_\alpha) \neq 0$.

Then we can define $h_\alpha = \dfrac{2H_\alpha}{(\alpha,\alpha)}$.

Note.

i) The 2 is to get an integer later.

ii) If $\mathfrak{g}$ is simple)(,) is unique up to scalar i.e. $(\ , \ )' = \xi(\ , \ ),\ \xi \in \mathbb{C}^\times$. Then

$$h'_\alpha = \frac{2H'_\alpha}{(\alpha,\alpha)'} = \frac{2\xi^{-1}H_\alpha}{\xi^{-2}\cdot\xi(\alpha,\alpha)} = h_\alpha$$

Hence, h_α is independent of (,).

34 April 4th, 2018

Today we discuss the connections between finite dimensional semi-simple Lie algebras over $\mathbb{C}$ and Root systems.

finite dimensional semi-simple Lie algebra over $\mathbb{C} \to$ Root systems $\big(\to$ Dynkin diagrams$\big)$¶¶

Structure of semi-simple Lie algebras

Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over $\mathbb{C}$, $(\cdot,\cdot)$ is the Killing form, $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$, and R root system of $\mathfrak{g}$.

Lemma 34.1 ($\mathfrak{sl}(2,\mathbb{C})$ triples)**.** *Let $\alpha \in R$. Pick $e_\alpha \in \mathfrak{g}_\alpha$ where $e_\alpha \neq 0$. Choose $f_\alpha \in \mathfrak{g}_\alpha$ by*

$$(e_\alpha, f_\alpha) = \frac{2}{(\alpha,\alpha)}.$$

(Recall by lemma 33.8 that $(\alpha,\alpha) \neq 0$.) Define $\mathfrak{sl}(2,\mathbb{C})_\alpha := \mathbb{C}e_\alpha \oplus \mathbb{C}f_\alpha \oplus \mathbb{C}h_\alpha$, which is a Lie subalgebra of $\mathfrak{g}$ isomorphic to $\mathfrak{sl}(2,\mathbb{C})$.

Proof. Claim: $[e_\alpha, f_\alpha] = (e_\alpha, f_\alpha)H_\alpha$. Indeed for all $h \in \mathfrak{h}$

$$\begin{aligned}(h, [e_\alpha, f_\alpha]) &= -([e_\alpha, h], f_\alpha)\\ &= ([h, e_\alpha], f_\alpha)\\ &= (\alpha(h)e_\alpha, f_\alpha)\\ &= (e_\alpha, f_\alpha)\cdot(H_\alpha, h)\\ &= ((e_\alpha, f_\alpha)H_\alpha, h)\\ &= (h, (e_\alpha, f_\alpha)H_\alpha).\end{aligned}$$

¶¶It turns out these maps are isomorphisms, but we save this for a later time.

Since the form is non-degenerate, the claim holds. Now $h_\alpha = \dfrac{2H_\alpha}{(\alpha,\alpha)}$, so

$$[e_\alpha, f_\alpha] = (e_\alpha, f_\alpha) \cdot \frac{(\alpha,\alpha)}{2} \cdot h_\alpha = h_\alpha.$$

Also,

$$[h_\alpha, e_\alpha] = \alpha(h_\alpha)e_\alpha = (H_\alpha, h_\alpha)e_\alpha = \frac{2(H_\alpha, H_\alpha)}{(\alpha,\alpha)} = 2e_\alpha.$$

Similarly $[h_\alpha, f_\alpha] = -2f_\alpha$. ▫

Lemma 34.2 (Lemma 6.43 in Kirillov)**.** *Let $\alpha \in R$. Then*

$$V = \mathbb{C}h_\alpha \oplus \bigoplus_{\substack{k\in\mathbb{Z}\\ k\neq 0}} \mathfrak{g}_{k\alpha} \subseteq \mathfrak{g}$$

is an irreducible subrepresentation of $\mathfrak{g}$ with respect to the adjoint action of $\mathfrak{sl}(2,\mathbb{C})_\alpha$ on $\mathfrak{g}$.

Proof. Check V is a subrepresentation of $\mathfrak{g}$

Example 34.3. $e_\alpha.\, \mathfrak{g}_{k\alpha} = [e_\alpha, \mathfrak{g}_{k\alpha}] \subseteq \mathfrak{g}_{(k+1)\alpha}$

Note that $V[0] = \mathbb{C}h_\alpha$ and $V[2k+1] = 0$ since $h_\alpha.x = 2kx$ for all $x \in \mathfrak{g}_{k\alpha}$. By Exercise 4.11 in the text, V is irreducible. ▫

Theorem 34.4 (Structure of Semi-simple Lie Algebras over $\mathbb{C}$)**.**

(1) *R spans $\mathfrak{h}^*$ as a $\mathbb{C}$-vector space and $\{h_\alpha\}_{\alpha\in R}$ span $\mathfrak{h}$ as a $\mathbb{C}$-vector space.*

(2) $\dim \mathfrak{g}_\alpha = 1$ *for all $\alpha \in R$.*

(3) *$\forall \alpha, \beta \in R$ then*

$$\beta(h_\alpha) \in \frac{2(\beta,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z}.$$

(4) *For any $\alpha \in R$, the reflection*

$$\begin{aligned} s_\alpha\colon \mathfrak{h}^* &\to \mathfrak{h}^* \\ \lambda &\mapsto \lambda - \lambda(h_\alpha)\alpha \\ &\quad\ \lambda - 2\frac{(\lambda,\alpha)}{(\alpha,\alpha)}\alpha \end{aligned}$$

preserves R, i.e. $s_\alpha(\beta) \in R$ $\forall \alpha, \beta \in R$. In particular, $-\alpha = s_\alpha(\alpha) \in R$ $\forall \alpha \in R$.

(5) *$\forall \alpha \in R$ $(\mathbb{C}\alpha) \cap R = \{\alpha, -\alpha\}$*

(6) *$\forall \alpha \in R$, $\forall \beta \in R$ such that $\beta \neq \pm\alpha$ then $V = \oplus_{k\in\mathbb{Z}}\, \mathfrak{g}_{\beta+k+\alpha}$ is an irreducible $\mathfrak{sl}(2,\mathbb{C})_\alpha$ representation with respect to the adjoint action.*

(7) *If $\alpha, \beta \in R$ such that $\alpha + \beta \in R$, then* $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$.

Proof. (1) Suppose $h \in \mathfrak{h}$ with $\alpha(h) = 0$ for all $\alpha \in R$. Then $\operatorname{ad} h\colon \mathfrak{g} \to \mathfrak{g}$ is the zero map. This implies $h \in \mathcal{Z}(\mathfrak{g}) = 0 \Rightarrow h = 0$. Thus R spans $\mathfrak{h}^*$. The final part comes from $\mathfrak{h} \leftrightarrow \mathfrak{h}^*$ by $\mathfrak{h}_\alpha \mapsto \alpha$.

(2) By Lemma 34.2 and representation theory for $\mathfrak{sl}(2, \mathbb{C})$, $V = \mathbb{C}h_\alpha \oplus \bigoplus \mathfrak{g}_{k\alpha}$ $\mathfrak{g}_{k\alpha}$ has $\dim 1$ for all $k, \alpha \Rightarrow k = 1$ for all $\alpha \in R$.

(3) $\beta(h_\alpha)$ is the weight of $x \in \mathfrak{g}_\beta$ with $\mathfrak{sl}(2, \mathbb{C})$-action $h_\alpha(x) = \beta(h_\alpha)x$. By representation theory of $\mathfrak{sl}(2, \mathbb{C})$, they are all integers.

(4) $x \in \mathfrak{g}_\beta$ has weight $n := \beta(h_\alpha)$ suppose $n > 0$, Then $f_\alpha^n\colon \mathfrak{g}_\beta \xrightarrow{\cong} \mathfrak{g}_{\beta-\alpha n}$, so if $0 \neq v \in \mathfrak{g}_\beta$ then $\mathfrak{g}_{\beta-n\alpha} \neq 0$. Which implies $\beta - n\alpha \in R$ with $s_\alpha(\beta) = \beta - n\alpha$. For $n < 0$ we use e_α^n instead. (5)-(7) Read yourselves. □

Example 34.5. $\mathfrak{g} = \mathfrak{sl}(3, \mathbb{C})$, $\mathfrak{h} = \mathbb{C}h_1 + \mathbb{C}h_2$ with

$$h_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \text{and} \qquad h_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Note that

$$[h_1, E_{12}] = 2E_{12} \qquad \text{and} \qquad [h_1, E_{12}] = -E_{12}$$

so $E_{12} \in \mathfrak{g}_\alpha$ with $\alpha(h_1) = 2$ and $\alpha(h_2) = -1$. Similarly $E_{23} \in \mathfrak{g}_\beta$ with $\beta(h_1) = -1$ and $\beta(h_2) = 2$. Now $E_{13} = [E_{12}, E_{23}] \in \mathfrak{g}_{\alpha+\beta}$. So $R = \{\pm\alpha, \pm\beta, \pm(\alpha+\beta)\}$ and $\begin{bmatrix} & \alpha & \alpha+\beta \\ -\alpha & & \beta \\ -\alpha-\beta & -\beta & \end{bmatrix}$ with $(\alpha, \beta) = (H_\alpha, H_\beta) = 2\cos(120°) = -1$.

35 April 6th, 2018

(Abstract) Root Systems

Definition. An *(abstract) root system* is a finite subset of a Euclidean space $R \subset E \backslash \{0\}$ where E is a Euclidean space, such that

(R1) $\operatorname{span}_{\mathbb{R}} R = E$;

(R2) $\forall \alpha, \beta \in R$: $n_{\alpha\beta} = \dfrac{2(\alpha, \beta)}{(\beta, \beta)} \in \mathbb{Z}$;

(R3) Let $s_\alpha\colon E \to E$ given by

$$s_\alpha(\lambda) = \lambda - \frac{2(\lambda, \alpha)}{(\alpha, \alpha)}\alpha.$$

Then $s_\alpha(\beta) \in R$ $\forall \alpha, \beta \in R$.

(R4) $\forall \alpha \in R$ $(\mathbb{R}\alpha) \cap R = \{\alpha, -\alpha\}$.

Note. s_α is the orthogonal reflection in the *root hyperplane* L_α:

$$L_\alpha = \alpha^\perp = \{\lambda \in E \mid (\lambda, \alpha) = 0\}.$$

Example 35.1. Let $\mathfrak{g}$ be a finite dimensional semi-simple Lie algebra over $\mathbb{C}$ and $\mathfrak{h} \subset \mathfrak{g}$ Cartan subalgebra. Then the set of roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$ is a root system is a root system in $\mathfrak{h}^*_{\mathbb{R}}$ where

$$\mathfrak{h}_{\mathbb{R}} = \mathrm{span}_{\mathbb{R}}\{h_\alpha \mid \alpha \in R\}$$

a real form of $\mathfrak{h}$ i.e. $\mathfrak{h}_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} = \mathfrak{h}$. $\mathfrak{h}^*_{\mathbb{R}} = \{\lambda \in \mathfrak{h}^* \mid \lambda(\mathfrak{h}_\alpha) \in \mathbb{R}\ \forall \alpha \in R\}$ (By a lemma $(\cdot, \cdot)$ Killing form restricted to $\mathfrak{h}^*_{\mathbb{R}}$ is a positive definite inner product.)

Notation. If $v \in V$, $\lambda \in V^*$ we define $\langle v, \lambda \rangle = \langle \lambda, v \rangle := \lambda(v)$.

Definition. Let $R \subset E$ be an (abstract) root system. The *coroot* $\check{\alpha}$ of $\alpha \in R$ is defined by $\check{\alpha} \in E^*$ where

$$\langle \check{\alpha}, \lambda \rangle = \frac{2(\lambda, \alpha)}{(\alpha, \alpha)}.$$

Note.

(1) This is consistent with the Lie algebra definition of coroot:

$$\check{\alpha} = h_\alpha = \frac{2H_\alpha}{(\alpha, \alpha)}$$

where $(H_\alpha, h) = \alpha(h)\ \forall h \in \mathfrak{h}$.

(2) Integrality says $\langle \alpha, \check{\beta} \rangle \in \mathbb{Z}\ \forall \alpha, \beta \in R$ and $s_\alpha(\lambda) = \lambda - \langle \lambda, \check{\alpha} \rangle \alpha$.

Example 35.2 (Root system of type A_{n-1})**.** Let $\{\varepsilon_i\}_{i=1}^n$ be the the orthonormal basis for $\mathbb{R}^n$. Let

$$\begin{aligned} E &= \{(\lambda_1, \dots, \lambda_n) \in \mathbb{R} \mid \sum_{i=1}^n \lambda_i = 0\} \\ R &= \{\varepsilon_i - \varepsilon_j \mid 1 \leqslant i, j \leqslant n,\ i \neq j\}. \end{aligned}$$

Then R spans E over $\mathbb{R}$. $(\varepsilon_i - \varepsilon_j, \varepsilon_i - \varepsilon_j) = 2 \Rightarrow (\alpha, \beta) \in \mathbb{Z} \quad \forall \alpha, \beta \in R \Rightarrow \dfrac{2(\alpha, \beta)}{(\beta, \beta)} = (\alpha, \beta) \in \mathbb{Z}$. Now,

$$\begin{aligned} s_{\varepsilon_i - \varepsilon_j}(\lambda_1, \dots, \lambda_i, \dots, \lambda_j, \dots, \lambda_n) &= (\lambda, \varepsilon_i - \varepsilon_j) \cdot (\varepsilon_i - \varepsilon_j) \\ &= \lambda - (\lambda_i - \lambda)(\varepsilon_i - \varepsilon_j) \\ &= \lambda - \lambda_i \varepsilon_i - \lambda \varepsilon_j + \lambda_i \varepsilon_j + \lambda_j \varepsilon_i \\ &= \lambda_1, \dots, \lambda_j, \dots, \lambda_i, \lambda_n \\ &\Rightarrow s_{\varepsilon_i - \varepsilon_j} \leftrightarrow (ij) \in S_n. \end{aligned}$$

Clearly $s_\alpha(\beta) \in R$. Lastly, $\mathbb{R}_\alpha \cap R = \{\pm\alpha\}$ is clear. Fact: A_{n-1} "is" the root system of $\mathfrak{sl}(n, \mathbb{C})$.

Definition. An *isomorphism* from $R_1 \subset E_1$ to $R_2 \subset E_2$ is a $\mathbb{R}$-linear isomorphism $\varphi\colon E_1 \to E_2$ such that

i) $\varphi(R_1) = R_2$;

ii) $n_{\varphi(\alpha)\varphi(\beta)} = n_{\alpha\beta}$ for all $\alpha, \beta \in R_1$.

Example 35.3. $R \subset E$ is isomorphic to $c.R$ a root system by $\varphi\colon E \to E$ by $\lambda \mapsto c \cdot \lambda$ for all $c \in \mathbb{R}\backslash\{0\}$.

Definition. The *Weyl group* $W = W(R)$ of a root system $R \subset E$ is the subgroup of $GL(E)$ generated by $\{s_\alpha \mid \alpha \in R\}$.

Lemma 35.4. *Let W be the Weyl group of a root system $R \subset E$.*

1) *W is finite;*

2) *$W \leqslant O(E)$ the orthogonal group*

3) *If $w \in W$, $\alpha \in R$ then $ws_\alpha w^{-1} = s_{w(\alpha)}$.*

Proof. 1) We know $W(R) \subseteq R$ since $s_\alpha(R) \subseteq R$, so we get a map $\varphi\colon W \to S_R$. We claim that φ is injective. Suppose $w \in \ker\varphi$, then $w(\alpha) = \alpha$ for all $\alpha \in R$. By R spanning E we get that $w = \mathrm{Id}_E$.

2) Each s_α is an orthogonal transformation (i.e, $(s_\alpha(\lambda), s_\alpha(\mu)) = (\lambda, \mu)$ for all λ, μ) $\Rightarrow$ $W \leqslant O(E)$

3) Consider the following calculation

$$\begin{aligned}
(ws_\alpha w^{-1})(\lambda) &= w(s_\alpha(w^{-1}(\lambda))) \\
&= w\left(w^{-1}(\lambda) - \frac{(w^{-1}(\lambda), \alpha)}{(\alpha, \alpha)}\alpha\right) \\
&= \lambda - 2\frac{(\lambda, w(\alpha))}{(w(\alpha), w(\alpha))}w(\alpha) \\
&= s_{w(\alpha)}(\lambda).
\end{aligned}$$

□

36 April 9th, 2018

More on (Abstract) Root Systems

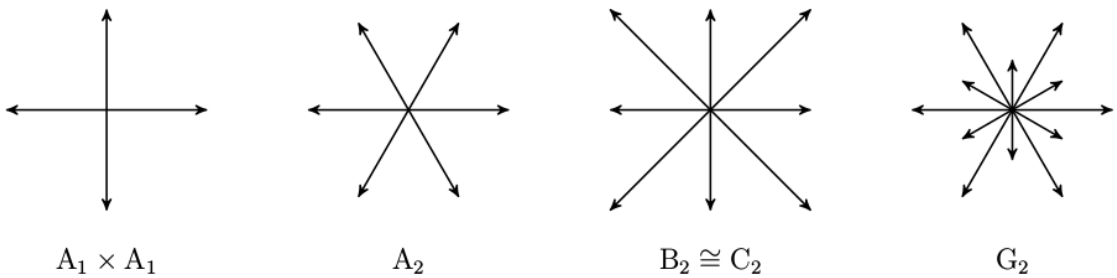
Theorem 36.1. *Let α, β be non-parallel roots of a root system R. Let $\theta \in (0, 2\pi)$ be the angle between them. Then $\theta \in \mathbb{Z}\frac{2\pi}{12} \cup \mathbb{Z}\frac{2\pi}{8}$ and $\pm\{n_{\alpha\beta}, n_{\beta\alpha}\} \in \big\{\{0,0\}, \{1,1\}, \{1,2\}, \{1,3\}\big\}$.*

Proof.

$$
\begin{aligned}
\mathbb{Z} \ni n_{\alpha\beta} n_{\beta\alpha} &= \frac{2(\alpha,\beta)}{(\beta,\beta)} \frac{2(\beta,\alpha)}{(\alpha,\alpha)} \\
&= \frac{4|\alpha|^2|\beta|^2 \cos^1\theta}{|\alpha|^2|\beta|^2} \\
&= 4\cos^2\theta
\end{aligned}
$$

□

Root Systems of Rank 2



$$
\begin{aligned}
A_1 \times A_1 \text{ (or } A_1 \sqcup A_1) &\leftrightarrow \mathfrak{sl}(2,\mathbb{C}) \times \mathfrak{sl}(2,\mathbb{C}) \\
A_2 &\leftrightarrow \mathfrak{sl}(3,\mathbb{C}) \\
B_2 = C_2 &\leftrightarrow \mathfrak{so}(5,\mathbb{C}) \cong \mathfrak{sp}(4,\mathbb{C}) \\
G_2 &\leftrightarrow \mathrm{Der}(\mathbb{O})
\end{aligned}
$$

Where $\mathrm{Der}(\mathbb{O})$ is the Lie algebra of derivations of the octonians, which is one of the five exceptional simple Lie algebras.

E_{reg}, Weyl chambers, and Positive Roots

Definition. Let $R \subset E$ be a root system. The set of *regular vectors* in E is

$$E_{\text{reg}} = \{\tau \in E \mid \forall \alpha \in R \colon (\tau,\alpha) \neq 0\} = E \backslash \bigcup_{\alpha \in R} L_\alpha \text{ where } L_\alpha = \alpha^\perp$$

The connected components of E_{reg} are called the *Weyl chambers* of R.

Given a Weyl chamber C, we associate a *polarization* of R:

$$R = R_+ \sqcup R_-, \text{ where } R_\pm = \{\alpha \in R \mid \pm(\alpha,\tau) > 0\},\ \tau \in \mathbb{C}$$

Call $\alpha \in R_+$ (respectively R_-) a *positive* (respectively *negative*) *root*.

Remark. This doesn't depend on the choice of $e \in C$. C, being defined as a set of linear inequalities of the form $(\alpha,\tau) > 0$ (or $(\alpha,\tau) < 0$), is intersection of (open) half-spaces. Therefore, C is convex. So if $\tau,\tau' \in C$ then the straight line segment $[\tau,\tau'] \subset C$. By the intermediate value theorem, $R_+^\tau = R_+^{\tau'}$.

Remark. If $\tau \in E_{\text{reg}}$ and $\alpha \in R$, then $\forall \beta \in R$:

$$(s_\alpha(\tau), \beta) = (\tau, s_\alpha(\beta)) \neq 0$$

Now $s_\alpha(\beta) \in R$, hence $s_\alpha(\tau) \in E_{\text{reg}}$. Moreover, if τ, τ' belong to the same connected component C of E_{reg}. Let $p = [0,1] \to E_{\text{reg}}$ be a path from τ to τ'. Then $s_\alpha \circ p$ is a path from $s_\alpha(\tau)$ to $s_\alpha(\tau')$. So W acts on $\pi_0(E_{\text{reg}}) = \{C \mid \text{ connected components}\}$. Lastly, note that, the Weyle group acts on R by root system automorphisms. (More generally we have $O(E) \to \text{Aut}(R)$).

Theorem 36.2. *The action of W on $\pi_0(E_{\text{reg}})$ is transitive. $\forall C, C' \in \pi_0(E_{\text{reg}})\ \exists w \in W: \ w(C) = C'$. Thus every root system has a unique polarization, up to a Weyl group automorphism of R.*

Proof. Let C, C' be any two Weyl chambers. Pick $\tau \in C$, $\tau' \in C'$ such that $\tau + \tau' \neq 0$. Then the line segment $[\tau, \tau']$ is contained in $E \backslash \{0\}$ and crosses some of the hyperplanes $L_{\beta_1}, L_{\beta_2}, \ldots, L_{\beta_k}$. Claim: If C_1 and C_2 are adjacent, i.e. $L_\beta = \text{span}_{\mathbb{R}}(\overline{C_1} \cap \overline{C_2})$, then $s_\beta(C_1) = C_2$. Pick $\tau \in C_1$, the line segment $[\tau, s_\beta(\tau)]$ only intersects L_β. Hence, $(s_\beta(\tau), \alpha)$ and (τ, α) have the same sign for all $\alpha \in R \backslash \{\beta\}$. Thus $C' = s_{\beta_k} \cdots s_{\beta_1}(C)$. □

37 April 11th, 2018

Simple Roots

Last time discussed how any root system R has a unique (up to Weyl group automorphism) polarization $R = R_+ \sqcup R_-$.

Definition. $\alpha \in R_+$ is *simple* if it is not a sum of two positive roots.

Lemma 37.1. *Every positive root is a sum of simple roots.*

Proof. Let $\alpha \in R$. If α is simple we are done. If not, $\alpha = \beta + \gamma$, for $\beta, \gamma \in R_+$. Then pick $\tau \in C_+ = \{\lambda \in E \mid (\lambda, \alpha') > 0\ \forall \alpha' \in R_+\}$. Then $(\alpha, \tau) = (\beta, \tau) + (\gamma, \tau)$ each of which is strictly greater of zero $\Rightarrow (\beta, \tau), (\gamma, \tau) < (\alpha, \tau)$. The set $\{(\alpha', \tau) \mid \alpha' \in R_+\}$ is finite; therefore, totally ordered. So we proceed by induction (or by contradiction). □

Proposition 37.2. *Every root is a unique combination of simple roots with integer coefficients:*

$$\alpha = \sum_{i=1}^{r} n_i \alpha_i, \quad n_i \in \mathbb{Z}$$

Where $\Pi = \{\alpha_i, \ldots, \alpha_r\}$ is the set of simple roots. Moreover, $\alpha \in R_\pm \Leftrightarrow \pm n_i \geqslant 0\ \forall i$

Proof. With out loss of generality $\alpha \in R_+$.

Step 1: Linear algebra fact: If $\{v_i\} \subset E$ such that $(v_i, V_j) \leqslant 0\ \forall i \neq j$, then $\{v_i\}$ is linearly independent. (Exercise)

Step 2 Suffice to show $(\alpha_i, \alpha_j) \leqslant 0 \; \forall i \neq j$. Let $\alpha, \beta \in \Pi$. We want to show that $(\alpha, \beta) \leqslant 0$. Suppose $(\alpha, \beta) > 0$. Then

$$\begin{aligned} s_\alpha(-\beta) &= -\beta - \langle -\beta, \check{\alpha} \rangle \alpha \\ &= -\beta + \langle \beta, \check{\alpha} \rangle \alpha \\ &= \frac{2(\beta, \alpha)}{(\alpha, \alpha)} > 0 \in R \end{aligned}$$

Aside. $R = R(\mathfrak{g})$ Since $\bigoplus_{k \in \mathbb{Z}} \mathfrak{g}_{-\beta + k\alpha}$ is an irreducible representation of $\mathfrak{sl}(2, \mathbb{C})$ and $\mathfrak{g}_{-\beta} \neq 0$. Thus $\mathfrak{g}_{-\beta + \langle \beta, \check{\alpha} \rangle \alpha} \neq 0 \Rightarrow -\beta, -\beta + \alpha, \ldots, -\beta + \langle \beta, \check{\alpha} \rangle \alpha$ This is a general fact, not just for $R(\mathfrak{g})$.

If $-\beta + \alpha \in R_+$, then $\beta + (-\beta + \alpha) = \alpha$ which is a contradiction to α simple. If instead $-\beta + \alpha \in R_-$, then $\beta - \alpha \in R_+$. So $\beta = \alpha + (\beta - \alpha)$ which contradicts β simple. So $(\alpha, \beta) \leqslant 0$. □

Exercise 37.3. If $\alpha, \beta \in R$, $\alpha \neq \pm\beta$, then $(\mathbb{Z}\alpha + \mathbb{Z}\beta) \cap R$ is a root system of rank 2.

Corollary 37.4. Π *is a basis for E over $\mathbb{R}$, so* $|\Pi| = \dim E = \operatorname{rank} R$.

Proof. Π linear independent and spans R over $\mathbb{Z}$, and R spans E over $\mathbb{R} \Rightarrow \Pi$ spans E over $\mathbb{R}$. □

Simple Reflections

Definition. $R = R_+ \sqcup R_-$, $\Pi = \{\alpha_1, \ldots, \alpha_r\}$. Then $s_i = s_{\alpha_i}$ are called *simple reflections*.

Proposition 37.5. *Any simple reflection s_i permutes the positive roots other than α_i, i.e. $s_i(R_+ \backslash \{\alpha_i\}) = R_+ \backslash \{\alpha_i\}$.*

Proof. Let $\beta \in R_+$ with $\beta = \sum_{j=1}^{\infty} n_j \alpha_j \; n_j \geqslant 0$. Then $s_i(\beta) = \beta - \langle < \beta, \alpha_i \rangle \alpha_i$. So if $s_i(\beta) \in R_-$, then $n_j \leqslant 0$ for all $j \neq i$. This implies that $n_j = 0$ for all $j \neq i$. Thus $\beta = n_i \alpha_i \;\Rightarrow\; \beta = \alpha_i$. □

Corollary 37.6. *The* Weyl vector, $\rho = \frac{1}{2} \sum_{\alpha \in R_+} \alpha$ *satisfies:*

$$\langle \rho, \check{\alpha}_i \rangle = 1 \qquad \forall i = 1, \ldots, r$$

equivalently $s_i(\rho) = \rho - \alpha_i$.

Theorem 37.7.

i) *The simple reflections generate the Weyl group.*

ii) $\forall \alpha \in R \; \exists w \in W$, $\alpha_i \in \Pi$: $\alpha = w(\alpha_i)$.

iii) W *acts simple transitively on $\pi_0(E_{\text{reg}})$ i.e. if $w(C) = C$, then $w = 1$.*

38 April 13th, 2018

Todays lecture is a discussion of $\mathfrak{sp}(4,\mathbb{C})$.

$\mathfrak{sp}(4,\mathbb{C})$

For $\mathbb{C}^4$ we have *symplectic form* $w\colon \mathbb{C}^4\times\mathbb{C}^4\to\mathbb{C}$, i.e. w is

- bilinear
- non-degenerate
- skew-symmetric

$\mathfrak{sp}(4,\mathbb{C}) = \{x\in\mathfrak{gl}(4,\mathbb{C}) \mid w(x.v,u)+w(v,x.u)=0\}$.

Theorem 38.1. *Over $\mathbb{C}$ all symplectic forms are equivalent (i.e coincide after a change of basis).*

We have matrix for w:

$$J = (w(\mathbb{e}_i,\mathbb{e}_j))_{i,j}$$

i.e., $w(a,b)=a^TJb$.

Example 38.2. A good choice $J=\begin{bmatrix}0 & I_2\\ -I_2 & 0\end{bmatrix}$ $\left(\text{or } \begin{bmatrix}0 & 1 & & \\ -1 & 0 & & \\ & & 0 & 1\\ & & -1 & 0\end{bmatrix}\right)$.

As such

$$\begin{aligned}
\mathfrak{sp}(4,\mathbb{C}) &= \{x \mid (x.v)^TJu + v^TJ(x.u) = 0\ \forall u,v\in\mathbb{C}^4\}\\
&= \{x \mid v^Tx^TJu + v^TJxu = 0\ \forall u,v\in\mathbb{C}^4\}\\
&= \{x \mid x^TJ + Jx = 0\}\\
&= \left\{x=\begin{bmatrix}A & B\\ C & D\end{bmatrix} \middle| \begin{bmatrix}A^T & C^T\\ B^T & D^T\end{bmatrix}\begin{bmatrix}0 & I_2\\ -I_2 & 0\end{bmatrix} + \begin{bmatrix}0 & I_2\\ -I_2 & 0\end{bmatrix}\begin{bmatrix}A & B\\ C & D\end{bmatrix}\right\}\\
&= \left\{x=\begin{bmatrix}A & B\\ C & D\end{bmatrix} \middle| -C^T+C=0,\ A^T+D=0,\ B^T-B=0\right\}\\
&= \left\{\left[\begin{array}{c|c} A & B=B^T\\ \hline C=C^T & -A^T\end{array}\right]\right\}
\end{aligned}$$

Now we claim we can generate a Cartan subalgebra $\mathfrak{h}$ by the following two matrices:

$$h_1 = \begin{bmatrix}1 & & & \\ & 0 & & \\ & & -1 & \\ & & & 0\end{bmatrix} \qquad h_2 = \begin{bmatrix}0 & & & \\ & 1 & & \\ & & 0 & \\ & & & -1\end{bmatrix}.$$

Now we construct matrices for generating the rest of $\mathfrak{sp}(4, \mathbb{C})$:

$$\begin{aligned}
F_{12} &= \begin{bmatrix} 0 & 1 & & \\ 0 & 0 & & \\ & & 0 & 0 \\ & & -1 & 0 \end{bmatrix}; \\
F_{21} &= F_{12}^T; \\
F_{13} &= \begin{bmatrix} 0 & 0 & 1 & 0 \\ & & & \\ & & & \\ & & & \end{bmatrix}; \\
F_{31} &= F_{13}^T; \\
F_{14} &= \begin{bmatrix} 0 & 0 & 0 & 1 \\ & & 1 & 0 \\ & & & \\ & & & \end{bmatrix}; \\
F_{41} &= F_{14}^T; \\
F_{24} &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ & & & 1 \\ & & & \\ & & & \end{bmatrix}; \\
F_{42} &= F_{24}^T.
\end{aligned}$$

Recall: $[E_{ij}, E_{k\ell}] = \delta_{jk}E_{i\ell} - \delta_{i\ell}E_{kj}$. Where E_{ij} are matrix units.

Note.

$$\begin{aligned}
[h_i, x^T] &= [x, h_i^T]^T \\
&= [x, h_i]^T \\
&= -[h_i, x]^T \\
&= -\alpha(h_i)x^T
\end{aligned}$$

$\Rightarrow$ $x^T \in \mathfrak{g}_{-\alpha}$ and $(\mathfrak{g}_\alpha)^T = \mathfrak{g}_{-\alpha}$, when $\mathfrak{h} \subseteq \{\text{diagonal matrices}\}$ (or $\{\text{symmetric}\}$).

Now we calculate the root decomposition.

$$\begin{aligned}
[h_1, F_{12}] &= [E_{11} - E_{33}, E_{12} - E_{43}] \\
&= E_{12} - E_{43} \\
&= 1 \cdot F_{12}; \\
[h_2, F_{12}] &= [E_{22} - E_{44}, E_{12} - E_{43}] \\
&= -E_{12} + E_{43} = (-1) \cdot F_{12}.
\end{aligned}$$

So $F_{12} \in \mathfrak{g}_\alpha$ and $F_{21} \in \mathfrak{g}_{-\alpha}$, where $\alpha(h_1) = 1$ and $\alpha(h_2) = -1$. Similarly,

$$\begin{aligned}
[h_1, F_{13}] &= [E_{11} - E_{33}, E_{13}] \\
&= 2 \cdot F_{13}; \\
[h_2, F_{12}] &= [E_{22} - E_{44}, E_{13}] \\
&= 0.
\end{aligned}$$

So $F_{13} \in \mathfrak{g}_\beta$ and $F_{31} \in \mathfrak{g}_{-\beta}$, where $\beta(h_1) = 2$ and $\beta(h_2) = 0$. By similar calculations we get

$$\begin{aligned}
&[h_1, F_{14}] = [E_{11} - E_{23}, E_{14} + E_{23}] = \cdots = F_{14}; \\
&[h_2, E_{14}] = F_{14};
\end{aligned}$$

with $F_{14} \in \mathfrak{g}_\gamma$ and $F_{41} \in \mathfrak{g}_{-\gamma}$, where $\gamma(h_1) = \gamma(h_2) = 1$, and

$$\begin{aligned}
[h_1, F_{24}] &= [E_{11} - E_{33}, E_{24}] \\
&= 0; \\
[h_2, F_{24}] &= [E_{22} - E_{44}, E_{24}] \\
&= 2 \cdot F_{24};
\end{aligned}$$

with $F_{24} \in \mathfrak{g}_\delta$ and $F_{42} \in \mathfrak{g}_{-\delta}$, where $\delta(h_1) = 0$ and $\delta(h_2) = 2$.

Note. The following relations hold for our roots: $\alpha + \delta = \gamma$, $\alpha + \gamma = 2\alpha + \delta = \beta$.

Thus $R = \pm\{\alpha, \delta, \gamma = \alpha + \delta, \beta = \delta + 2\alpha\}$. Using the trace form on $\mathfrak{h}_\mathbb{R} = \mathbb{R}h_1 \oplus \mathbb{R}h_2$, we have $(h_i, h_j) = \mathrm{Tr}(h_i h_j) = 2\delta_{ij}$. By rescaling we define $(h_i, h_j) := \delta_{ij}$

Now we work to find the coroots. $H_\alpha = a_1 h_1 + a_2 h_2$, by the definition of coroots we know that

$$\begin{aligned}
1 &= \alpha(h_1) = (H_\alpha, h_1) = a_1 \\
-1 &= \alpha(h_2) = (H_\alpha, h_2) = a_2.
\end{aligned}$$

This implies that $H_\alpha = h_1 - h_2 \Rightarrow \check{\alpha} = h_\alpha = \dfrac{2H_\alpha}{(H_\alpha, H_\alpha)} = H_\alpha = h_1 - h_2$. Similarly, $H_\delta = 2h_2$ implies $\check{\delta} = \dfrac{2H_\delta}{(H_\delta, H_\delta)} = \dfrac{1}{2} H_\delta = h_2$.

Now we compute $n_{\delta\alpha}$ and $n_{\alpha\delta}$:

$$\begin{aligned}
n_{\delta\alpha} &= \frac{2(\delta, \alpha)}{(\alpha, \alpha)} = \delta(\check{\alpha}) = \delta(h_1 - h_2) = -2 \\
n_{\alpha\delta} &= \alpha(\check{\delta}) = \alpha(h_2) = -1.
\end{aligned}$$

Note. $(\delta, \delta) = 2(\alpha, \alpha)$.

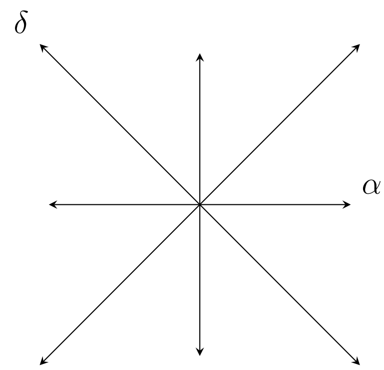


Figure 16: Root System of $\mathfrak{sp}(4,\mathbb{C})$

So

$$\begin{aligned}4\cos^2\theta &= n_{\alpha\delta}n_{\delta\alpha}\\ &= (-1)(-2)\\ \cos^2\theta &= \frac{1}{2}.\end{aligned}$$

Now $(\alpha,\delta)<0$ implies that $\cos\theta<0$. Hence, $\cos\theta=\frac{-1}{\sqrt{2}}\Rightarrow\theta=3\cdot\frac{2\pi}{8}$. A suitable choice for R_+ would be $R_+=\{\alpha,\delta,\delta+\alpha,\delta+2\alpha\}$ and $\Pi=\{\alpha,\delta\}$.

The Weyl group $W=\langle s_1=s_\alpha,s_2=s_\delta\rangle$ such that $s_1s_2=\rho_{\frac{2\pi}{4}}$, so $W\cong D_4$ the dihedral group of order 8.

The Cartan matrix (which will be discussed more next time) is:

$$\begin{bmatrix}2 & -1\\ -2 & 2\end{bmatrix}.$$

39 April 16th, 2018

Cartan Matrices & Dynkin Diagrams

Let R root system $\subset E$; R_+ choice of positive roots; $\Pi\subset R_+$ simple roots $\Pi=\{\alpha_1,\cdots,\alpha_r\}$.

Theorem 39.1. *R can be recovered from Π and the numbers $n_{\alpha\beta}\in\Pi$.*

Proof. Recall $R=W(\Pi)$ and W is generated by the simple relfections (by Theorem 37.7). So if $\alpha\in R$, then $\exists i_1,\dots,i_k,j\in\{1,\dots,r\}$ such that $\alpha=s_{i_k}\cdots s_{i_2}s_{i_1}(\alpha_j)$.

Note. $s_{i_1}(\alpha_j)=\alpha_j-\langle\alpha_j,\check{\alpha}_{i_1}\rangle\alpha_{i_1}=\alpha_j-(n_{\alpha_j\alpha_{i_1}})\alpha_{i_1}$.

By linearity, $s_{i_2}s_{i_1}(\alpha_j)$ can be computed from $n_{\alpha_j\alpha_{i_2}},n_{\alpha_j\alpha_{i_1}},n_{\alpha_{i_1}\alpha_{i_2}}$. By induction α can be computed using only knowledge of Π and $n_{\alpha\beta}$ for $\alpha,\beta\in\Pi$. □

Definition. The *cartan matrix* of R is defined by: $A = (a_{ij})_{i,j=1}^r$ where

$$a_{ij} n_{\alpha_j \alpha_i} = \langle \alpha_j, \check{\alpha}_i \rangle = \frac{2(\alpha_j, \alpha_i)}{(\alpha_i, \alpha_i)}.$$

Corollary 39.2. *R is uniquely determined by A.*

Example 39.3. We look at the cartan matrices for rank 2 root systems.

$$A_1 \sqcup A_1 : \quad \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$A_2 : \quad \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

$$B_2 = C_2 : \quad \begin{bmatrix} 2 & -1 \\ -2 & 2 \end{bmatrix}$$

$$G_2 : \quad \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$$

Note.

i) $a_{ii} = 2$ for all i

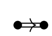
ii) $(a_{ij}, a_{ji}) \in \{(0,0) \cup (\{-1,-2,-3\} \times \{-1\}) \cup (\{-1\} \times \{-1,-2,-3\})\}$

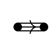
iii) $|a_{ij}| < |a_{ji}| \Leftrightarrow |\alpha_j| > |\alpha_i|$.

Rank 2 Dynkin diagrams

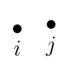
$A_1 \sqcup A_1$

A_2

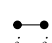
B_2 

G_2 

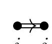
Definition. The *Dynkin diagram* D of R is a graph with vertex set Π (often identified with $\{1, \dots, r\}$) and edges of the following four kinds:



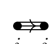
no-edge $a_{ij} = a_{ji} = 0$



$a_{ij} = a_{ji} = -1$



$(a_{ij}, a_{ji}) = (-2, -1)$



$(a_{ij}, a_{ji}) = (-3, -1)$

<u>Main point:</u> A, hence R, can be recovered from D.

Definition. A set $S \subset E$ is the *orthogonal union* of two subsets $S_1, S_2 \subseteq S$ if

1) $S = S_1 \cup S_2$
2) $u \perp v \; \forall u \in S_1, v \in S_2$.

Notation: $S = S_1 \biguplus_{\perp} S_2$

Definition. $S \subset E$ *decomposable* if $S = S_1 \biguplus_{\perp} S_2$, $S_i \neq 0$.

40 April 18th, 2018

Classification of Root Systems

Lemma 40.1. *Let R be a root system, Π a set of simple roots, A the cartan matrix, and D the Dynkin diagram. TFAE:*

1) *R is* irreducible*, i.e. $R = R_1 \biguplus_{\perp} R_2 \Rightarrow R_1 = \varnothing$ or $R_2 = \varnothing$.*
2) *Π is irreducible.*
3) *D is connected.*
4) *A is* indecomposable*, i.e. it cannot be written as a block diagonal matrix $A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}$ even after reordering Π.*

Proof. Bonus Homework Question. □

Lemma 40.2. *If R is a reducible subset of E: $R = R_1 \biguplus_{\perp} R_2$ with $R_i \neq 0$, then R_i are root systems in $E_i = \operatorname{span} R_i$.*

Proof. Bonus Homework Question. □

Corollary 40.3. *Without loss of generality, we may assume R is irreducible.*

Thus our goal should be to describe all connected Dynkin diagrams!

Coxeter Graphs

Definition. A *coxeter graph* $\Gamma = (\Gamma_0, \Gamma_1)$ is an undirected loopless graph such that each edge $e \in \Gamma_1$ has multiplicity $m_e \in \mathbb{Z}_{>0}$. By thinking of no-edge as multiplicity 0, we may think of

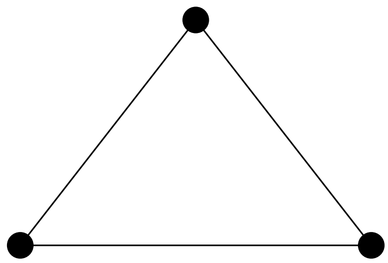
$$\Gamma_1 \subseteq (\mathbb{Z}_{\geqslant 0})^{\binom{\Gamma_0}{2}}$$

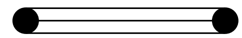
and the adjacency mattrix for Γ

$$A = \begin{bmatrix} 0 & & a_{ij} \\ & \ddots & \\ a_{ji} & & 0 \end{bmatrix}$$

is symmetric with $a_{ij} \in \mathbb{Z}_{\geqslant 0}$.

Example 40.4. Below are two examples of coxeter graphs.





Example 40.5. Forgetting the direction of a Dynkin diagram gives a coxeter-graph.

Admissible Sets

Definition. An *admissible set* $U = \{e_1, \dots, e_r\} \subseteq E$ is a set of linearly independent unit vectors such that $\forall i \neq j$:

1) $\cos \varphi_{ij} = (e_i, e_j) \leqslant 0$;

2) $4(e_i, e_j)^2 \in \{0, 1, 2, 3\}$.

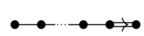
Example 40.6. The following will be our main example to consider $U = \left\{ \frac{\alpha}{|\alpha|} \mid \alpha \in \Pi \right\}$.

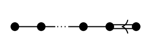
To any admissible set U, we can attach a coxeter graph $\Gamma = (\Gamma_0, \Gamma_1)$ where $\Gamma_0 = U$ and e_j, e_i are connected by an edge of multiplicity $4(e_i, e_j)^2$. So we have the following commuting diagram:

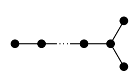
$$\begin{tikzcd} \Pi \arrow[r, rightsquigarrow] \arrow[d, rightsquigarrow] \arrow[dr, phantom, "\circlearrowleft"] & U \arrow[d, rightsquigarrow] \\ D \arrow[r, rightsquigarrow] & \Gamma \end{tikzcd}$$

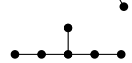
Theorem 40.7 ((Classification of Root systems). *Let R be any root system. Then its Dynkin diagram D is a union of diagram of the following type:*

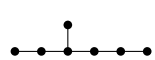
A_r $(r \geqslant 1)$

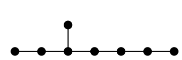
B_r $(r \geqslant 2)$ 

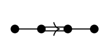
C_r $(r \geqslant 3)$ 

D_r $(r \geqslant 4)$ 

E_6 

E_7 

E_8 

F_4 

G_2 

Where types A through D are considered the classical types. Moreover, any two of these correspond to non-isomorphic root systems.

Proof. We show that any connected admissible graph is one the above types (with orientation removed). Let $U = \{e_1, \dots, e_r\}$ be any admissible set, and Γ be its coexeter graph. Then:

1) Any subset of an admisslbe set is admissible.

Proof of 1). Clear. □

2) $c := |\{\{e_i, e_j\} \mid i \neq j \text{ and } e_i, e_j \text{ connected}\}|$. Then $c < r$.

Proof of 2). Let $e = \sum e_i$. Then

$$0 < (e, e) = r + \sum_{i<j} 2(e_i, e_j) \leqslant r + (-c).$$

Hence, $c < r$. □

3) Γ has no cycles.

Proof of 3). If $\Gamma' \subseteq \Gamma$ is a cycle, by 1) Γ' is an admissible graph, but this contradicts 2) because in cycles $c = r$. □

4) The degree of any vertex (with multiplicity) is $\leqslant 3$.

Proof of 4). Suppose $e \in U$ has edges to $\eta_1, \eta_2, \dots, \eta_k$, then $4(e, \eta_i)^2 \in \{1, 2, 3\}$. By 3) $(\eta_i, \eta_j) = 0$ $\forall i \neq j$ (else Γ would contain a cycle). $W = \text{span}\{\eta_i\}$ $e' := \text{proj}_W e = \sum_{i=1}^k (e, \eta_i)\eta_i$. Now $e' \neq e$ since $\{e, \eta_1, \dots, \eta_k\}$ are linearly independent. Hence,

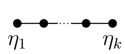
$$4 = 4 \cdot (e, e) > 4(e', e') = \sum_{i=1}^k 4(e, \eta_i)^2 = \deg e.$$

□

5) By 4) the only possible diagrams containing a triple edge is

.

6) If $\{\eta_1, \dots, \eta_k\} \subset U$ with their induced subgraph is



then they can be replaced by a single point, i.e. $(U \backslash \{\eta_1, \dots, \eta_k\}) \cup \{\eta\}$, where $\eta = \sum_{i=1}^k \eta_i$ is, is admissible. □

41 April 20th, 2018

Classification Proof Continued

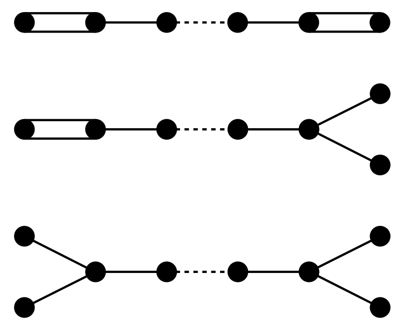
Proof of Theorem 40.7 Con't. 6) Linear roots can be deformed to a single point. $U' = (U \backslash \{\eta_1, \dots, \eta_k\}) \cup \{\eta\}$ with $\eta = \sum_{i=1}^k \eta_i$.

Proof of 6). Clearly, U' is linearly independent. Now

$$(\eta, \eta) = (\sum \eta_i, \sum \eta_i) = k + (k-1)\cdot(-1) = k - k + 1 = 1$$

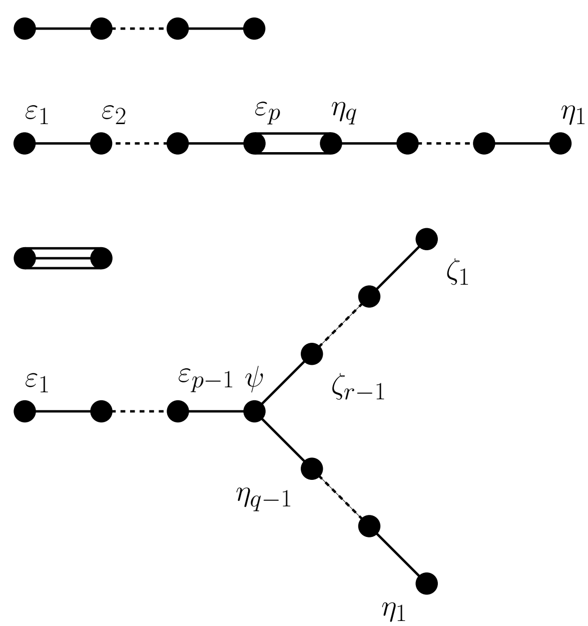
We want to show that $4(\varepsilon, \eta)^2 \in \{0, 1, 2, 3\}$ for all $\varepsilon \in U'$. Now any $\epsilon \in U' \backslash \eta$ is connected to at most one of the $\{\eta_1, \ldots, \eta_k\}$ (else there would be a cycle), so $(\varepsilon, \eta) = 0$ or $(\varepsilon, \eta) = (\varepsilon, \eta_i)$ for exactly one i. Hence, $4(\varepsilon, \eta)^2 = 4(\epsilon, \eta_i)^2 \in \{0, 1, 2, 3\}$. ▫

7) Γ contain no subgraphs of the form:



Proof of 7). If we collapse lines to points as in 6), then we would reach a vertex of degree greater than 3 from each of these graphs which is a contradiction to 4). ▫

8) Any connected Γ of an admissible set must be of the form: 9) The only graphs of the



second variety are

(type F_4)
&
(type BC_n)

Proof of 9). Define $\varepsilon := \sum_{i=1}^{p} i\varepsilon_i$ and $\eta := \sum_{j=1}^{q} j\eta_j$. Then

$$(\varepsilon,\varepsilon) = \sum_1^p i^2 - \sum_1^{p-1} i(i+1) = \frac{p(p+1)}{2}$$

Similarly,

$$(\eta,\eta) = \frac{q(q+1)}{2}.$$

Now

$$(\varepsilon,\eta)^2 = p^2q^2(\varepsilon_p,\eta_q)^2 = \frac{p^2q^2}{2}$$

By Cauchy-Schwarz:

$$\frac{p^2q^2}{2} = (\varepsilon,\eta)^2 < (\varepsilon,\varepsilon)(\eta,\eta) = \frac{p(p+1)}{2}\frac{q(q+1)}{2}$$

This implies $(p-1)(q-1) < 2$, so either $p = 1$ or $q = 1$ (Type BC_n), or $p = q = 2$ (Type F_4). ▫

10) The only Γ of the fourth type is D_n or E_6, E_7, E_8.

Proof of 10). Define $\varepsilon := \sum_{i=1}^{p} i\varepsilon_i$, $\eta := \sum_{j=1}^{q} j\eta_j$, and $\zeta := \sum \ell = 1\ell\zeta_\ell$. Then ε, η, and ζ are necessarily orthogonal and linearly independent, and $\psi \notin \operatorname{span}\{\varepsilon,\eta,\zeta\}$ as in 4) $\cos^2\theta_\varepsilon + \cos^2\theta_\eta + \cos^2\theta_\zeta < 1$ where these are the angles between the projection of $\operatorname{proj}_{\operatorname{span}\{\varepsilon,\eta,\zeta\}}\psi$ and the respective roots. On the other hand,

$$\cos^2\theta_\varepsilon = \frac{(\varepsilon,\psi)^2}{|\epsilon|^2} = \frac{(p-1)(\varepsilon_{p-1},\psi)}{|\varepsilon|^2} = \frac{1}{2}(1-\frac{1}{p}).$$

Similarly for θ_η and θ_ζ. This implies that

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1$$

With out loss of generality, $p \geqslant q \geqslant r$. If $r = 1$ we get A_n, else $r = 2$. Then $2 \leqslant q < 4$. As such we get $(p,2,2)$ for type D_n, $(3,3,2),(4,3,2),(5,3,2)$ are E_6, E_7, E_8 respectively. ▫

This completes the proof. ▫

Serre's Theorem

We have seen that

$$\{\text{f.d. s.s. Lie alg}/\mathbb{C}\}_{/\text{iso}} \xrightarrow{\mathcal{D}} \{\text{not necessarily connected Dynkin diagrams}\}$$

by

$$\mathfrak{g} \overset{\text{Choose C.S.A}}{\longmapsto} R \overset{\text{Choose } t\in E_{\text{reg}}}{\longmapsto} R_+ \mapsto \Pi \overset{\text{ordering}}{\longmapsto} A \mapsto D.$$

Serre's Theorem $\mathcal{D}$ is bijective by constructing an explicit inverse $D \mapsto \mathfrak{g}(D)$.

Theorem 41.1 (Chevalley-Serre)**.** *Let $\mathfrak{g}$ be as above, and D its Dynkin Diagram, $A =$ cartan matrix (with respect to some order on vertices). Then $\mathfrak{g}$ is generated by a subset $\{e_i, f_i, h_i\}_{i=1}^r$ satisfying*

$$\begin{array}{ll} \textit{chevalley 1940's} & \begin{cases} [e_i, f_j] = \delta_{ij} h_i \\ [h_i, e_j] = a_{ij} e_j \\ [h_i, f_j] = a_{ij} f_j \end{cases} \\ \textit{Serre 1960's} & \begin{cases} \operatorname{ad}(e_i)^{1-a_{ij}}(e_j) = 0 \\ \operatorname{ad}(f_i)^{1-a_{ij}}(f_j) = 0 \end{cases} \end{array}$$

Theorem 41.2 (Serre's Theorem)**.** *Let D be a Dynkin diagram and A its cartan matrix. Let $\mathfrak{g}(D)$ be the free Lie algebra on $3r$ symbols $\{e_i, f_i, h_i\}_{i=1}^r$ module the Chevalley-Serra relations.*

Then $\mathfrak{g}(D)$ is a finite dimensional semi-simple Lie algebra over $\mathbb{C}$ with Dynkin diagram D. Moreover, $\mathfrak{g} \cong \mathfrak{g}(D)$ for any $\mathfrak{g}$ whose Dynkin diagram is D.

42 April 23th, 2018

Highest Weight Theory

Motivation: Recall the representation V_n of $\mathfrak{sl}(2, \mathbb{C})$:

$$\begin{aligned} V_n &= \operatorname{span}\{x^n, x^{n-1}y, \ldots, y^n\} \\ \rho(e) &= x\partial_y \\ \rho(f) &= y\partial_x \\ \rho(h) &= x\partial_x - y\partial_y \end{aligned}$$

V_n contains a special vector $v_0 = x^n$ with three properties:

1) v_0 is a *weight vector*:

$$h.v_0 = (x\partial_x - y\partial_y)(x^n) = nx^n = nv_0.$$

2) V_0 is a *highest weight vector*:

$$e.v_0 = (x\partial_y)(x^n) = 0.$$

3) V_n is generated by v_0:

$$V_n = \operatorname{span}_{\mathbb{C}}\{a_1 \cdot (a_2 \cdot (\cdots (a_k \cdot v_0)) \cdots) \mid k \geqslant 0 \ a_i \in \mathfrak{sl}(2, \mathbb{C})\}$$

equivalently, using that

$$a_1 \cdot (a_2 \cdot (\cdots (a_k \cdot v_0)) \cdots) = (a_1 a_2 \cdots a_k) \cdot v_0$$

where $(a_1 a_2 \cdots a_k) \in U(\mathfrak{sl}(2, \mathbb{C}))$, $V_n = U(\mathfrak{sl}(2, \mathbb{C})).v_0$.

Definition. Let $\mathfrak{g}$ be a finite dimensional semi-simple Lie algebra over $\mathbb{C}$, choose a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$, and $R_+ \subset R$ a choice of positive roots. A representation of $\mathfrak{g}$ $V = (V, \rho)$ is a *highest weight representation* if $\exists v_0 \in V$ and $\lambda \in \mathfrak{h}^*$ such that

1. v_0 is a *weight vector* of weight λ:
$$h.v_0 = \lambda(h)v_0 \quad \forall h \in \mathfrak{h}\,.$$

2. v_0 is a *highest weight vector*:
$$\mathfrak{n}_+.v_0 = 0$$
where $\mathfrak{n}_+ = \oplus_{\alpha\in R_+} \mathfrak{g}_\alpha$ i.e. $e.v_0 = 0$ $\forall e \in \mathfrak{g}_\alpha$, $\forall \alpha \in R_+$.

3. V is generated by v_0
$$V = U(\mathfrak{g}).v_0$$

Theorem 42.1. *Any finite dimensional irreducible representation of $\mathfrak{g}$ is a highest weight representation.*

Proof. $\{\rho(h) \mid h \in \mathfrak{h}\}$ is a family of commuting linear operators on V, hence there is at least one common eigenvector, w, say. Let $\mu \in \mathfrak{h}^*$ be defined by $\rho(h)w = \mu(h)w$. So $V' = \oplus_{\xi\in\mathfrak{h}^*} V_\xi$ is a non-zero subspace of V, where $V_\xi = \{v \in V \mid \rho(h)v = \xi(h)v\}$. So $w \in V_\mu$. But V' is actually a subrepresentation:
$$\mathfrak{g}_\alpha \cdot V_\xi \subseteq V_{\xi+\alpha}.$$
V irreducible implies that $V = V'$.

Let $P(V) = \{\xi \in \mathfrak{h}^* \mid V_\xi \neq 0\}$ be the support of V. Choose $h \in \mathfrak{h}$ such that $\langle\alpha, h\rangle > 0$ $\forall \alpha \in R_+$ (e.g. $h = \check{\tau}$, where τ defines R_+). Then let $\lambda \in P(V)$ be such that $\langle\lambda, h\rangle$ is maximal. Then $\langle\lambda+\alpha, h\rangle > \langle\lambda, h\rangle$ for all $\alpha \in R_+$. This implies that $\forall \alpha \in R_+$ $\lambda+\alpha \notin P(V)$. Hence, $\mathfrak{g}_\alpha V_\lambda \subseteq V_{\lambda+\alpha} = 0$ for all $\alpha \in R_+$.

Let v_0 be any nonzero vector in V_λ. Then 1) and 2) hold. that v_0 generates V is obvious since V is irreducible. □

Proposition 42.2. *If V is a finite dimensional irreducible representation of $\mathfrak{g}$ of highest weight λ, then $\lambda(\check{\alpha}_i) \in \mathbb{Z}_{\geqslant 0}$ for ever simple coroot $\check{\alpha}_i = h_i$.*

Proof. Consider the action of $\mathfrak{sl}(2,\mathbb{C})_{\alpha_i} \subset \mathfrak{g}$ on V. Let $v_0 \in V$ be a highest weight vector. Consider
$$V_i = \oplus_{k\in\mathbb{Z}_{\geqslant 0}} V_{\lambda-k\alpha_i}.$$
This is an $\mathfrak{sl}(2,\mathbb{C})_{\alpha_i}$-subrepresentation of V. v_0 satisfies $h_i v_0 = \lambda(h_i)v_0$, $e_i.v_0 = 0$ and v_0 generates a finite dimensional $\mathfrak{sl}(2,\mathbb{C})_{\alpha_i}$-representation.

Let N be minimal such that $f_i^N.v_0 = 0$. Then
$$0 = e_i.(f_i^N.v_0) = (f_i^N e_i).v_0 + [e_i, f_i^N].v_0 = \cdots = (\lambda(h_i) - (N-1))f_i^{N-1}.v_0$$

Thus $\lambda(h_i) = N - 1$, so we are done. □

So we have a map

$$\{\text{f.d. irreps of } \mathfrak{g}\}_{/iso.} \xrightarrow{\Phi} P_+$$

where $P_+ = \{\lambda \in \mathfrak{h}^* \mid \lambda(\check{\alpha}_i \in \mathbb{Z}_{\geqslant 0}\}$ are the *dominant integral weights.*

Goal: Show that we can go back (i.e. there exists an inverse of Φ).

Plan:

1) To any $\lambda \in \mathfrak{h}^*$ construct a universal highest weight representation $M(\lambda)$ of highest weight λ. (Verma Module)

2) Each $M(\lambda)$ has a unique irreducible quotient $L(\lambda)$.

3) Show that $L(\lambda)$ is finite dimensional iff $\lambda \in P_+$.

4) $L(\lambda) \cong L(\mu) \iff \lambda = \mu$.

43 April 25th, 2018

Highest Weight Theory Part II

Recall: A *highest weight representation* of $\mathfrak{g}$ is a representation generated by a *highest weight vector*:

1) $hv_0 = \lambda(h)v_0$ for all $h \in \mathfrak{h}$

2) $ev_0 = 0$ for all $e \in \mathfrak{g}_\alpha$ and $\alpha \in R_+$

3) $V = U(\mathfrak{g})v_0$

Definition. Let $\lambda \in \mathfrak{h}^*$. The corresponding *Verma module* $M(\lambda)$ is defined by

$$M(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}1_\lambda$$

where $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}_+ = \mathfrak{h} \oplus_{\alpha \in R_+} \mathfrak{g}_\alpha$, and $\mathbb{C}1_\lambda$ is the 1-dimensional representation of $\mathfrak{b}$ given by:

$$\begin{aligned} h.1_\lambda &= \lambda(h)1_\lambda \quad && \forall h \in \mathfrak{h} \\ x.1_\lambda &= 0 && \forall x \in \mathfrak{n}_+ \end{aligned}$$

and $U(\mathfrak{g})$ is regarded as a right $U(\mathfrak{b})$ module.

Aside on Tensor Products

Let R be a ring, M a right R-module, and N a left R-module. Then $M \otimes_R N$ is the abelian group generated by the symbols $m \otimes n$ subject to the relations:

1) $\mathbb{Z}$ bilinearity

$$\begin{aligned}(m_1 + m_2) \otimes n &= m_1 \otimes n + m_2 \otimes n \\ m \otimes n_1 + n_2 &= m \otimes n_1 + m \otimes n_2\end{aligned}$$

2) R-balanced

$$(m.r) \otimes n = m \otimes (r.m)$$

Moreover, if M is a (S, R)-bimodule, then $M \otimes_R N$ is a left S-module via $s.(m \otimes n) = s.m \otimes n$.

$M(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}1_\lambda$ with $\mathbb{C}1_\lambda$ is a left $U(\mathfrak{b}$-module and $U(\mathfrak{g})$ is a $(U(\mathfrak{g}), U(\mathfrak{b}))$-bimodule.

Example 43.1. $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$, $\mathfrak{b} = \mathbb{C}h \oplus \mathbb{C}e$, $\mathbb{C}1_\lambda$ for $\lambda \in \mathfrak{h}^* \cong \mathbb{C}$ such that $h.1_\lambda = \lambda(h)1_\lambda = \lambda' 1_\lambda$ and $e.1_\lambda$. Below we see how we can simplify elements in $M(\lambda)$.

$$\begin{aligned}M(\lambda) \ni hef \otimes 1_\lambda &= h(fe + [e, f]) \otimes 1_\lambda \\ &= (hfe + h^2) \otimes 1_\lambda \\ &= hfe \otimes 1_\lambda + h^2 \otimes 1_\lambda \\ &= hf \otimes 0 + 1 \otimes h^2 1_\lambda \\ &= 1 \otimes (\lambda^1)^2 1_\lambda \\ &= (\lambda')^2 (1 \otimes 1_\lambda)\end{aligned}$$

<u>Recall:</u> The PBW theorem (Thm 23.1) says that $U(\mathfrak{sl}(2, \mathbb{C}))$ has a basis

$$\{f^k h^\ell, e^m \mid k, \ell, m \geqslant 0\}$$

So

$$M(\lambda) = \sum_{k,\ell,m \geqslant 0} \mathbb{C} f^k h^\ell e^m \otimes 1_\lambda = \sum_{k,\ell \geqslant 0} \mathbb{C} f^k \otimes h^\ell 1_\lambda = \sum_{k \geqslant 0} \mathbb{C} f^k \otimes 1_\lambda$$

Theorem 43.2. *$M(\lambda) \cong U(\mathfrak{n}_-)$ as left $U(\mathfrak{n}_-)$-modules*

Proof. $\varphi \colon U(\mathfrak{n}_-) \to M(\lambda)$ by $x \mapsto x \otimes 1_\lambda$. φ is a surjective using the PBW theorem (thm 23.1):

$$\begin{aligned}M(\lambda) &= U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \otimes_{U(\mathfrak{b})} \mathbb{C}1_\lambda \\ &= U(\mathfrak{n}_-) U(\mathfrak{b}) \otimes_{U(\mathfrak{b})} \mathbb{C}1_\lambda \\ &= U(\mathfrak{n}_-) \otimes \mathbb{C}1_\lambda \\ &\subseteq \operatorname{im} \varphi.\end{aligned}$$

φ is injective: By PBW theorem $U(\mathfrak{g})$ is free as a right $U(\mathfrak{b})$-module on a basis for $u(\mathfrak{n}_-)$: $U(\mathfrak{g}) \cong U(\mathfrak{n}_-) \otimes_{\mathbb{C}} U(\mathfrak{b})$ by properties of tensors one can show that

$$\begin{aligned} U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}1_\lambda &\cong (U(\mathfrak{n}_-) \otimes_{\mathbb{C}} U(\mathfrak{b})) \otimes_{U(\mathfrak{b})} \mathbb{C}1_\lambda \\ &= U(\mathfrak{n}_-) \otimes_{\mathbb{C}} (U(\mathfrak{b}) \otimes_{U(\mathfrak{b})} \mathbb{C}1_\lambda) \\ &\cong U(\mathfrak{n}_-) \otimes_{\mathbb{C}} \mathbb{C}1_\lambda \\ &\cong U(\mathfrak{n}_-). \end{aligned}$$

□

Corollary 43.3. *The support of $M(\lambda)$ is*

$$P(M(\lambda)) = \lambda - Q_+ = \{\lambda - \sum_{i=1}^{r} k + i\alpha_i \mid k_i \in \mathbb{Z}_{\geqslant 0}\}$$

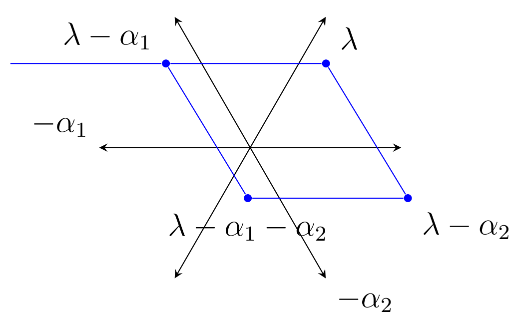
where $\{\alpha_1, \ldots, \alpha_r\} = \Pi$ the set of simple roots. $Q = \mathbb{Z}R = \oplus_1^r \mathbb{Z}\alpha_i$ and $Q_+ = \sum_1^r \mathbb{Z}_{\geqslant 0}\alpha_i$

Proof. $f_{\beta_1}^{k_1} \cdots f_{\beta_n}^{k_n} \otimes 1_\lambda$ where $R_+ = \{\beta_1, \ldots, \beta_n\}$. Then

$$\begin{aligned} h.(f_{\beta_1}^{k_1} \cdots f_{\beta_n}^{k_n} \otimes 1_\lambda) &= h f_{\beta_1}^{k_1} \cdots f_{\beta_n}^{k_n} \otimes 1_\lambda \\ &= (f_{\beta_1}^{k_1} \cdots f_{\beta_n}^{k_n} h + [h, f_{\beta_1}^{k_1} \cdots f_{\beta_n}^{k_n}]) \otimes 1_\lambda \\ &\overset{(*)}{=} \lambda(h) f_{\beta_1}^{k_1} \cdots f_{\beta_n}^{k_n} \otimes 1_\lambda - (k_1\beta_1 + \cdots + k_n\beta_n)(h) f_{\beta_1}^{k_1} \cdots f_{\beta_n}^{k_n} \otimes 1_\lambda \\ &= (\lambda - (k_1\beta_1 + \cdots + k_n\beta_n))(h) f_{\beta_1}^{k_1} \cdots f_{\beta_n}^{k_n} \otimes 1_\lambda. \end{aligned}$$

Where $(*)$ is by $[h, f_{\beta_i}] = -\beta_i(h) f_{\beta_i}$ for every i. □

Example 43.4. $\mathfrak{sl}(3, \mathbb{C})$ $\lambda = k_1 w_1 + k_2 w_2$ where $w_i \in \mathfrak{h}^*$ and $w_i(h_j) = \delta_{ij}$ which are the *fundamental weights*. Where the blue lattice is $P(M(\lambda))$.



44 April 27th, 2018

Goal: Classify all finite dimensional irreducible represents of a finite dimensional semi-simple Lie algebra over $\mathbb{C}$ $\mathfrak{g}$, (V, ρ).

Verma Modules

$$M(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}1_\lambda$$

Proposition 44.1.

i) *$M(\lambda)$ is a highest weight representation of $\mathfrak{g}$ of highest weight λ.*

ii) *Every highest weight representation of $\mathfrak{g}$ of highest weight λ is a quotient of $M(\lambda)$.*

iii) $M(\lambda)_\lambda = \mathbb{C}(1 \otimes 1_\lambda)$

iv) $P(M(\lambda)) = \lambda - Q_+ = \{\lambda - \sum k_i\alpha_i \mid k_i \in \mathbb{Z}_{\geqslant 0}\}$

v) *$M(\lambda)$ has a unique maximal (proper) submodule $N(\lambda)$. Hence $M(\lambda)$ has a unique irreducible quotient $V(\lambda) = L_\lambda = M(\lambda)/N(\lambda)$.*

Proof. i) Put $v_\lambda = 1 \otimes 1_\lambda \in M(\lambda)$,

$$\begin{aligned} h.v_\lambda = h.(1 \otimes 1_\lambda) &= h1 \otimes 1_\lambda \\ &= h \otimes 1_\lambda \\ &= 1 \otimes h.1_\lambda \\ &= 1 \otimes \lambda(h)1_\lambda \\ &= \lambda(h)(1 \otimes 1_\lambda). \end{aligned}$$

This implies v_λ is a weight vector of weight λ. Also $\forall e \in \mathfrak{n}_+$,

$$e.v_\lambda = e \otimes 1_\lambda = 1 \otimes 0 = 0$$

because $\mathfrak{n}_+ \subset \mathfrak{b}$. So v_λ is a highest weight vector.

$$U(\mathfrak{g})v_\lambda = U(\mathfrak{g}).(1 \otimes 1_\lambda) = U(\mathfrak{g}) \otimes 1_\lambda = M(\lambda).$$

ii) Let W be any highest weight representation of $\mathfrak{g}$ of highest weight λ. $w_\lambda \in W$ be a (non-zero) highest weight vector of weight λ.

Consider the map

$$\begin{aligned} \psi \colon U(\mathfrak{g}) \times \mathbb{C}1_\lambda &\to W \\ (a, \xi 1_\lambda) &\mapsto \xi a.w_\lambda \end{aligned}$$

Then

- ψ is $\mathbb{Z}$-bilinear (biadditive)

- Write $U(\mathfrak{b}) = U(\mathfrak{n}_+)U(\mathfrak{h})$. We show that $b \in U(\mathfrak{b})$ $\psi(ab, \xi 1_\lambda = \psi(a, b.\xi 1_\lambda)$. If $b = h \in \mathfrak{h}$, then

$$\begin{aligned}\psi(ah, \xi 1_\lambda) &= \xi a h 1_\lambda \\ &= \xi a \lambda(h) 1_\lambda \\ &= \lambda(h) \xi a 1_\lambda \\ &= \psi(a, \lambda(h)\xi 1_\lambda) \\ &= \psi(a, h.\xi 1_\lambda).\end{aligned}$$

  Similarly for $\mathfrak{b} = e \in \mathfrak{n}_+$. So ψ induces a map from $M(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}1_\lambda \xrightarrow{\tilde{\psi}} W$. Since W is generated by w_λ, the map $\tilde{\psi}$ is surjective. So $W \cong \dfrac{M(\lambda)}{\ker \tilde{\psi}}$.

v Let $N(\lambda) := \sum\limits_{\substack{S \subsetneq M(\lambda) \\ \text{subrepresentations}}} S$ We want to show that $N(\lambda) \subsetneq M(\lambda)$.

<u>Facts:</u> Any subrepresentation of a weight representation is a weight representation. This implies all S have a weight decomposition $S = \oplus_{\mu \in \lambda = Q_+} S_\mu$ and so does $N(\lambda)$:

$$N(\lambda) = \oplus_{\mu \in \lambda - Q_+} N(\lambda)_\mu.$$

Since each $S \subseteq M(\lambda)$ $S_\lambda = 0$. $(S_\lambda \subseteq M(\lambda)_\lambda \subset \mathbb{C}v_\lambda)$. Hence $N(\lambda)_\lambda = \oplus_{S \subsetneq M(\lambda)} S_\lambda = 0$.

Thus $N(\lambda)$ is proper subrepresentation. □

To summarize:

- Every finite dimensional irreducible representation is a finite dimensional irreducible highest weight representation.
- Every irreducible highest weight representation is $\cong V(\lambda$ for some $\lambda \in \mathfrak{h}^*$.
- Question: is when is $\dim V(\lambda) < \infty$?

Note. If $\lambda, \mu \in \mathfrak{h}^*$ $\lambda \neq \mu$ is $V(\lambda \cong V(\mu)$? False.

Suppose $\varphi \colon V(\lambda) \to V(\mu)$ is an isomorphism. Then $\varphi(V_\lambda)$ is a highest weight vector of λ generating $V(\mu)$. This contradicts $V(\lambda)$ is the unique irreducible quotient of $M(\lambda)$.

So $\{V(\lambda) \mid \lambda \in \mathfrak{h}^*\}$ is a complete set of representatives for isoclasses of irreducible highest weight representations of $\mathfrak{g}$.

Theorem 44.2. $\dim V(\lambda) \leqslant \infty$ *iff* $\lambda \in P_+$ *i.e.* $\lambda(\check{\alpha}_i) \in \mathbb{Z}_{\geqslant 0}$ $\forall i = 1, \dots, r$.

Proof. $(\Rightarrow)$: Last time using $\mathfrak{sl}(2, \mathbb{C})_{\alpha_i}$.
$(\Leftarrow)$: Hard. □

Characters

Let $V = \oplus_{\lambda \in \mathfrak{h}^*} V_\lambda$. Then we define the following, $\operatorname{ch} V := \sum (\dim V_\lambda) e^\lambda \in \mathbb{Z}[\mathfrak{h}^*]$ with $e^\lambda e^\mu = e^{\lambda+\mu}$ and $e^0 = 1$.

Theorem 44.3.

1) $\operatorname{ch}(V \oplus W) = (\operatorname{ch} V) + (\operatorname{ch} W)$

2) $\operatorname{ch}(V \otimes W) = (\operatorname{ch} V)(\operatorname{ch} W)$

3) *If V, W are finite dimensional, then $V \cong W$ iff $\operatorname{ch} V = \operatorname{ch} W$.*

Theorem 44.4 (Weyl Character Formula)**.** $\lambda \in P_+$ *then*

$$\operatorname{ch}(V(\lambda)) = \frac{\sum\limits_{w \in W} (-1)^{\ell(w)} e^{w(\lambda+\rho)}}{\prod_{\alpha \in R_+} (e^{\alpha/2} - e^{-\alpha/2})}$$

where $\rho = \dfrac{1}{2} \sum\limits_{\alpha \in R_+} \alpha$