Worksheet 1: Ring quotients and operations

1. Consider the subset $2\mathbb{Z}+1$ of $\mathbb{Z}$ consisting of all odd integers. Define a relation on $\mathbb{Z}$ by $a \sim b$ iff $a - b \in 2\mathbb{Z}+1$.

   (a) Is $\sim$ an equivalence relation on $\mathbb{Z}$? Prove it or find a counter-example. If yes, continue on to (b), otherwise skip to the next problem.

   (b) Let $\mathbb{Z}/(2\mathbb{Z}+1)$ denote the set of equivalence classes $[a]$ for $a \in \mathbb{Z}$. Trying to define $[a]+[b] = [a+b]$, do we get a well-defined operation?

   (c) Trying to define $[a] \cdot [b] = [ab]$, do we get a well-defined operation?

2. Consider the subset $\mathbb{Z} \subset \mathbb{Q}$. Define a relation $\sim$ on $\mathbb{Q}$ by $a \sim b$ iff $a-b \in \mathbb{Z}$.

   (a) Is $\sim$ an equivalence relation on $\mathbb{Q}$? Prove it or find a counter-example. If yes, continue on to (b), otherwise skip to the next problem.

   (b) Let $\mathbb{Q}/\mathbb{Z}$ denote the set of equivalence classes $[a]$. Trying to define $[a]+[b] = [a+b]$, do we get a well-defined operation?

   (c) Trying to define $[a] \cdot [b] = [ab]$, do we get a well-defined operation?

3. Repeat the above steps for the following subsets J of a ring R:

   (i) $R = \mathbb{Q} \times \mathbb{Q}$ and $J = \mathbb{Q} \times \{0\}$

   (ii) $R = \mathbb{Q} \times \mathbb{Q}$ and $J = \Delta = \{(p, q) \mid p, q \in \mathbb{Q}\}$

   (iii) $R = \mathbb{Z}/6\mathbb{Z} = \{0, 1, 2, 3, 4, 5\}$ and $J = \{0, 2, 4\}$

   (iv) $R = \mathbb{Q}[x]$ and $J = \{a_2x^2 + a_4x^4 + \cdots + a_{2k}x^{2k} \mid a_i \in \mathbb{Q},\ k \in \mathbb{Z}_{\geq 0}\}$