2023 Fall Math 301 Final Exam **Name**: ______________________________

last first

Instructions: This exam has a total of 100 points. You have 120 minutes. You must show all your work to receive full credit. If you use any result from a theorem or corollary in the book, you need to state it clearly. For the results from the examples and exercises, you have to give the details of the proofs. The points attached to each problem are indicated beside the problem. You are not allowed to use any book, notes, or calculator.

Question	Points
1	/12
2	/12
3	/13
4	/13
5	/ 13
6	/12
7	/13
8	/12
Total	/100

1. (12 points) Suppose a, b, c are positive integers such that $a|c$ and $b|c$. Let $d = \gcd(a, b)$. Prove that $ab|cd$.

2. (12 points) Let a, b, n be integers. Prove that the equation $[a]x = [b]$ has a solution in $\mathbb{Z}_n$ if and only if $(a, n)|b$.

3. (13 points) Let R be a ring such that $a^2 = a$ for every $a \in R$. Prove the following properties. (Hint: Consider $(a+b)^2$.)

   (a) (6 points) $a + a = 0_R$ for all $a \in R$.

   (b) (7 points) $ab = ba$ for all $a, b \in R$.

4. (13 points)

(a) (7 points) Determine if $f(x) = x^3 + x + 1$ is irreducible in $\mathbb{Z}_5[x]$. Be sure to explain your reasoning.

(b) (6 points) Construct a field with 125 elements. You need to explain why your contruction works.

5. (13 points) Determine if each of the following additive group is cyclic. If it is cyclic, find a generator for the group. If not, explain why.

   (a) (7 points) $G = \mathbb{Z}_3 \times \mathbb{Z}_4$.

   (b) (6 points) $G = \mathbb{Z}_3 \times \mathbb{Z}_6$.

6. (12 points) Let $f : \mathbb{Z} \to U_7$ be given by $f(n) = [3^n]$. Prove that f is a surjective group homomorphism .

7. (13 points) Consider the permutation $\sigma = (45678)(234)(123) \in S_8$.

(a) (5 points) Write σ as a product of disjoint cycles.

(b) (5 points) Write σ as a product of transpositions.

(c) (3 points) Determine if σ is even or odd.

8. (12 points) Let G be the group of units in $\mathbb{Z}/20\mathbb{Z}$, and $H = \langle 3 \rangle$. Find $|G|$, $|H|$ and $[G : H]$.

(Extra page for calculations)