5.28. Let I, J be ideals of a ring R. Suppose that the function

$$f : R \longrightarrow R/I \times R/J$$
$$f(r) = (r+I, r+J)$$

is surjective. Prove that $I + J = (1)$.

Sol. Since f is surj, there is $r \in R$ such that

$$f(r) = (1+I, 0+J).$$

By definition of f, this means

$$(r+I, r+J) = (1+I, 0+J)$$

Equivalently $1-r \in I$, $r \in J$

Thus $1 = (1-r) + r \in I+J$.

Since $I+J$ is an ideal, therefore $(1) \subseteq I+J$.

Since $(1)=R$, $I+J \subseteq (1)$ is trivial. Hence $(1) = I+J$. ∎

5.6 Let R, S be rings and I an ideal of R, J an ideal of S. Prove $I \times J$ is an ideal of $R \times S$.

Sol. $I \times J$ is nonempty; Since I and J are ideals we know $0_R \in I$ and $0_S \in J$. Therefore $(0_R, 0_S) \in I \times J$, So $I \times J$ is nonempty.

$I \times J$ closed under $+$: Let $x, y \in I \times J$. Then

$x = (a, b)$ $y = (c, d)$

for some $a, c \in I$, $b, d \in J$.

Thus

$$x + y = (a, b) + (c, d) = (a + c, b + d)$$

Since I is an ideal, $a + c \in I$, Since J is an ideal, $b + d \in J$, So $x + y \in I \times J$. $\longrightarrow$

$I \times J$ has <u>absorption property</u>:

Let $x \in R \times S$, $y \in I \times J$

WTS $xy \in I \times J$ and $yx \in I \times J$.

We have $x = (r, s)$ and $y = (a, b)$ for some $r \in R$, $s \in S$, $a \in I$, $b \in J$.

So

$xy = (r, s)(a, b) = (ra, sb)$

Since I is an ideal of R, $ra \in I$. Since J is an ideal of S, $sb \in J$. Thus $xy \in I \times J$.

Similarly $yx = (a, b)(r, s) = (ar, bs)$ and $ar \in I$, $bs \in J$ since I, J ideals of R, S respectively.

This proves $I \times J$ is an ideal of $R \times S$.

7.21 Prove if $q \in \mathbb{Z}$ is irreducible then q is irreducible in $\mathbb{Z}[x]$.

<u>Sol</u> If $q = f(x)g(x)$ for some non-units $f(x), g(x) \in \mathbb{Z}[x]$, then taking degree on both sides, (using $\mathbb{Z}$ is integral domain)

$$0 = \deg f(x) + \deg g(x)$$

But $\deg f(x) \geq 0$, $\deg g(x) \geq 0$ ($\mathbb{Z}[x]$ integral domain and $q \neq 0$).

So both $f(x)$ and $g(x)$ are constants, a_0 and b_0, say.

$$q = a_0 b_0$$

Since a_0, b_0 are non-units in $\mathbb{Z}[x]$, they are non-units in $\mathbb{Z}$. This contradicts that q is irreducible in $\mathbb{Z}$.

7.29 $f(x) = 7x^3 + 6x^2 + 4x + 6 \in \mathbb{Q}[x]$

Prove $f(x)$ is irreducible in $\mathbb{Q}[x]$ in 3 ways.

Sol 1 Eisenstein's Crit. with $p=2$

$$2 \nmid 7, \quad 2 \mid 6, 4, 6, \quad 2^2 \nmid 6$$

So $f(x)$ is irred. in $\mathbb{Q}[x]$.

Sol 2 Suppose $f(x)$ is reducible in $\mathbb{Q}[x]$. Since $\deg f(x) \leq 3$, $f(x)$ has a rational root $c = p/q$ where $p, q \in \mathbb{Z}$, $\gcd(p,q) = 1$, $p > 0$.
By Rational Root Test (since $f \in \mathbb{Z}[x]$) $p \mid 6$ and $q \mid 7$. Check

$$p/q \in \pm\{1, 2, 3, 6, 1/7, 2/7, 3/7, 6/7\}$$

None are roots of $f(x)$ (I'm skipping details here) contradiction

Sol 3 Reduce mod $p = 5$:

$$\overline{f(x)} = 2x^3 + x^2 + 4x + 1 \in \frac{\mathbb{Z}}{5\mathbb{Z}}[x]$$

Check $\overline{f(x)}$ no roots in $\frac{\mathbb{Z}}{5\mathbb{Z}}$.

therefore, since $\deg \underline{f(x)} \le 3$, $\underline{f(x)}$ is irred. in $\frac{\mathbb{Z}}{5\mathbb{Z}}[x]$.

By Reduction Modulo $p=5$, $f(x)$ is irreducible in $\mathbb{Z}[x]$ hence by Gauss's Lemma $f(x)$ irr in $\mathbb{Q}[x]$.

__7.30__ $g(x) = x^5 + x^4 + x - 1$

Prove $g(x)$ irred. in $\mathbb{Q}[x]$ using Eisenstein's Crit.

__Sol.__ No p applies. Try shift

$g(x+1) = (x+1)^5 + (x+1)^4 + x =$

$= x^5 + 5x^4 + 10x^3 + 10x^2 + 5x + 1$

$+ \quad x^4 + 4x^3 + 6x^2 + 4x + 1 \quad + x$

$= x^5 + 6x^4 + 14x^3 + 16x^2 + 10x + 2$

$p=2$ works:

$2 \nmid 1$, $2 \mid 6, 14, 16, 10, 2$, $2^2 \nmid 2$

So $g(x)$ is irreducible by Eistenstein