Study Guide for Exam 1 – MATH 3010 Fall 2024

Chapter 1.

- Well-Ordering Principle
- Division with Remainder
- Divisibility, greatest common divisor, Euclid's algorithm
- Primes and Irreducible integers, Fundamental Theorem of Arithmetic

Chapter 2.

- Congruence classes $[a]_n$, the set $\mathbb{Z}/n\mathbb{Z}$, the operations $+$ and $\cdot$ in $\mathbb{Z}/n\mathbb{Z}$
- $[a]_n$ is invertible (has a multiplicative inverse) iff $\gcd(a,n)=1$
- $\mathbb{Z}/n\mathbb{Z}$ is an integral domain iff n is prime (this includes 0)
- $\mathbb{Z}/n\mathbb{Z}$ is a field iff n is irreducible (this excludes 0)
- Fermat's Little Theorem: $a^{p-1} \equiv 1 (\text{mod } p)$ if p is a prime not dividing a

Chapter 3.

- Definition of a Ring; Examples $\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}, \mathbb{R}[x], M_{2,2}(\mathbb{R})$
  (a) If R is a ring then the set of 2×2-matrices $M_{2,2}(R)$ is a ring.
  (b) If R is a ring then the set of polynomials $R[x]$ is a ring.
- Basic Properties: 0_R, 1_R are unique, additive inverse $-a$ is unique, additive cancellation, $0_R a = 0_R = a 0_R$ for all $a \in R$.
- Definitions of zero-divisors and invertible elements; integral domain and field
- Field$\Rightarrow$Integral Domain$\Rightarrow$Commutative$\Rightarrow$Ring

Sample problems.

1. Find all elements in the ring $\mathbb{Z}/18\mathbb{Z}$ that are (a) zero-divisors (b) invertible
2. (a) Find $m, n \in \mathbb{Z}$ such that $53m + 4n = 1$. (b) Find the inverse of $[4]_{53}$ in $\mathbb{Z}/53\mathbb{Z}$.
3. Find all invertible elements of $M_{2,2}(\mathbb{Z}/2\mathbb{Z})$.
4. Let n be a positive integer and s the digit sum of n. (For example if $n = 5344$ then $s = 5+3+4+4 = 16$.) Prove that $n \equiv s \pmod 9$.
5. An element a of a ring R is *nilpotent* if $a^n = 0$ for some positive integer n. Show that if a is nilpotent then $1-a$ is invertible. (*Hint:* Think about geometric series.)
6. Show that every element of $\mathbb{Z}/n\mathbb{Z}$ is either invertible or a zero-divisor.
7. Suppose R is an integral domain. Prove that the following cancellation law holds: If $a, b, c \in R$, $a \neq 0_R$ and $ab = ac$, then $b = c$.
8. Let $\mathbb{R}_{>0}$ denote the set of positive real numbers. Define two operations $\oplus$ and $\odot$ on $\mathbb{R}_{>0}$ as follows:
$$x \oplus y = xy, \qquad x \odot y = x^y$$
   Is $\left(\mathbb{R}_{>0}, \oplus, \odot\right)$ a ring? Why/why not?