Solution for Math 301 Homework 1

Appendix B

9. Let $A = \{1, 2, 3, 4\}$. Exhibit functions f and g from A to A such that $f \circ g \neq g \circ f$.

Define $f,\ g : A \to A$ by $f(a) = 1$ for $a = 1,\ 2,\ 3,\ f(4) = 2$ and $g(a) = 4$ for all $a \in A$. Then

$$g \circ f(a) = 4 \text{ and } f \circ g(a) = 2 \text{ for all } a \in A.$$

11. Is the subset B closed under the given operation?

(b) $B =$ odd integers; operation: addition in $\mathbb{Z}$.

Not closed. $1 \in B$ but $1 + 1 = 2 \notin B$.

(d) $B =$ odd integers; operation $*$ on $\mathbb{Z}$, where $a*b$ is defined to be the number $ab-(a+b)+2$.

Closed. If $a,\ b \in B$, then a and b are odd. So, $a = 2m+1,\ b = 2n+1$ for some integers m and n. Hence,

$$a * b = ab - (a + b) + 2 = (a - 1)(b - 1) + 1 = (2m)(2n) + 1 \text{ is odd.}$$

12. Find the image of the function f when

(b) $f : \mathbb{Z} \to \mathbb{Q}$; $f(x) = x - 1$.

For every $x \in \mathbb{Z}$, $f(x) = x - 1 \in \mathbb{Z}$.

Suppose $y \in \mathbb{Z}$. Let $x = y + 1 \in \mathbb{Z}$. Then $f(x) = x - 1 = y$.

Therefore, image of $f = \mathbb{Z}$.

(c) $f : \mathbb{R} \to \mathbb{R}$; $f(x) = -x^2 + 1$.

For every $x \in \mathbb{R}$, $f(x) = -x^2 + 1 \in \mathbb{R}$ and $-x^2 + 1 \leqslant 1$.

Suppose $y \in \mathbb{R}$ and $y \leqslant 1$. Let $x = \sqrt{1 - y} \in \mathbb{R}$. Then $f(x) = -x^2+1 = -(1-y)+1 = y$.

Therefore, image of $f = \{y \in \mathbb{R} : y \leqslant 1\}$.

24. Determine whether the given operation on $\mathbb{R}$is commutative (that is, $a * b = b * a$ for all $a,\ b$) or associative (that is, $a * (b * c) = (a * b) * c$ for all $a,\ b,\ c$).

(a) $a * b = 2^{ab}$.

(b) $a * b = a * b^2$.

(d) $a * b = (a + b)/2$.

(f) $a * b = b$.

	a	b	d	f
Commutative	Yes	No	Yes	No
Associative	No	No	No	Yes

Explanation:

(a) For all $a, b \in \mathbb{R}$, $a * b = 2^{ab} = 2^{ba} = b * a \Rightarrow *$ is commutative.

$1*(1*3) = 1*(2^{1\cdot 3}) = 1*8 = 2^{1\cdot 8} = 2^8$, $(1*1)*3 = 2^{1\cdot 1}*3 = 2*3 = 2^{2\cdot 3} = 2^6 \neq 1*(1*3) \Rightarrow *$ is not associative.

(b) $1 * 2 = 1 \cdot 2^2 = 4$, $2 * 1 = 2 \cdot 1^2 = 2 \neq 1 * 2 \Rightarrow *$ is not commutative.

$1 * (1 * 3) = 1 * (1 \cdot 3^2) = 1 * 9 = 1 \cdot 9^2 = 81$, $(1 * 1) * 3 = 1 * 3 = 1 \cdot 3^2 = 9 \neq 1 * (1 * 3) \Rightarrow *$ is not associative.

(d) For all $a, b \in \mathbb{R}$, $a * b = (a + b)/2 = (b + a)/2 = b * a \Rightarrow *$ is commutative.

$1 * (1 * 3) = 1 * ((1 + 3)/2) = 1 * 2 = (1 + 2)/2 = 3/2$, $(1 * 1) * 3 = ((1 + 1)/2) * 3 = 1 * 3 = (1 + 3)/2 = 2 \neq 1 * (1 * 3) \Rightarrow *$ is not associative.

(f) $1 * 2 = 2$, $2 * 1 = 1 \neq 1 * 2 \Rightarrow *$ is not commutative.

For all $a, b, c \in \mathbb{R}$, $a * (b * c) = a * c = c$, $(a * b) * c = b * c = c = a * (b * c) \Rightarrow *$ is associative.

25. Prove that the given function is injective.

(b) $f : \mathbb{R} \to \mathbb{R}$; $f(x) = x^3$. For $a,\ b \in \mathbb{R}$,

$$f(a) = f(b) \Rightarrow a^3 = b^3 \Rightarrow a = (a^3)^{1/3} = (b^3)^{1/3} = b.$$

Therefore, f is injective.

(d) $f : \mathbb{R} \to \mathbb{R}$; $f(x) = -3x + 5$. For $a,\ b \in \mathbb{R}$,

$$f(a) = f(b) \Rightarrow -3a + 5 = -3b + 5 \Rightarrow -3a = -3b \Rightarrow a = b.$$

Therefore, f is injective.

26. Prove that the given function is surjective.

(c) $f : \mathbb{R} \to \mathbb{R}$; $f(x) = -3x + 5$.

For $y \in \mathbb{R}$, let $x = \dfrac{5 - y}{3} \in \mathbb{R}$. Then $f(x) = -3\left(\dfrac{5 - y}{3}\right) + 5 = -(5 - y) + 5 = y$.

Therefore, f is surjective.

(d) $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Q}$; $f(a, b) = a/b$ when $b \neq 0$ and 0 when $b = 0$.

For $y \in \mathbb{Q}$, $y = a/b$ for some $a,\ b \in \mathbb{Z}$, with $b \neq 0$. Then $(a, b) \in \mathbb{Z} \times \mathbb{Z}$ and $f(a, b) = a/b = y$.

Therefore, f is surjective.

28.

(a) Let $f : B \to C$ and $g \to C \to D$ such that $g \circ f$ is injective. Prove that f is injective.

Suppose the contrary that f is not injective. Then there exist x and y in B such that $x \neq y$ and $f(x) = f(y) \Rightarrow g(f(x)) = g(f(y)) \Rightarrow g \circ f$ is not injective, a contradiction.

Therefore, f is injective.

(b) Give an example of the situation in part (a) in which g is not injective.

Let $B = \{1\} = D$ and $C = \{-1,\ 1\}$. Define $f : B \to C$ by $f(b) = b$ and $g : C \to D$ by $g(c) = c^2$. Then $g(1) = 1 = g(-1)$ and g is not injective but $g \circ f : B \to D$ is injective.

1.1

2. Find the quotient q and remainder r when a is divided by b, without using technology. Check your answers.

(a) $-51 = (-9)6 + 3,\ q = -9,\ r = 3.$

(b) $302 = (15)19 + 17,\ q = 15,\ r = 17.$

9. Prove that the cube of any integer a has to be exactly one of these forms:$9k$ or $9k + 1$ or $9k + 8$ for some integer k. [*Hint:* By the Division Algorithm, a must be of the form $3q$ or $3q + 1$ or $3q + 2$.]

Let $a = 3q + r$, where $q,\ r \in \mathbb{Z}$ and $0 \leqslant r < 3$. Then

$$\begin{aligned} a^3 &= (3q + r)^3 \\ &= (3q)^3 + 3(3q)^2 r + 3(3q)r^2 + r^3 \\ &= 27q^3 + 27q^2 r + 9qr^2 + r^3 \\ &= 9(3q^3 + 3q^2 r + qr^2) + r^3 \end{aligned}$$

Let $k = (3q^3 + 3q^2 r + qr^2)$. We have

$$\begin{aligned} r = 0 \Rightarrow a^3 &= 9k \\ r = 1 \Rightarrow a^3 &= 9k + 1 \\ r = 2 \Rightarrow a^3 &= 9k + 8 \end{aligned}$$