Math 301 Homework 6 (due on October 11, 2023)

Section 3.3

12. Which of the following functions are homomorphisms?

(a) $f : \mathbb{Z} \to \mathbb{Z}$, defined by $f(x) = -x$.

(b) $f : \mathbb{Z}_2 \to \mathbb{Z}_2$, defined by $f(x) = -x$.

(d) $h : \mathbb{R} \to M(\mathbb{R})$, defined by $h(a) = \begin{pmatrix} -a & 0 \\ a & 0 \end{pmatrix}$.

(e) $f : \mathbb{Z}_{12} \to \mathbb{Z}_4$, defined by $f([x]_{12}) = [x]_4$, where $[u]_n$ denotes the class of the integer u in $\mathbb{Z}_n$.

19. Show that $S = \{0, 4, 8, 12, 16, 20, 24\}$ is a subring of $\mathbb{Z}_{28}$. Then prove that the map $f : \mathbb{Z}_7 \to S$, defined by $f\left([x]_7\right) = [8x]_{28}$ is a homomorphism.

30. Let $f : R \to S$ be a homomorphism of rings and let $K = \{r \in R | f(r) = 0_S\}$. Prove that K is a subring of R.

42. If $(m, n) \neq 1$. Prove that $\mathbb{Z}_{mn}$ is not isomorphic to $\mathbb{Z}_m \times \mathbb{Z}_n$.

Section 4.1

5. Find polynomials $q(x)$ and $r(x)$ such that $f(x) = g(x)q(x) + r(x)$ and $r(x) = 0$ or $\deg r(x) < \deg\left(g(x)\right.$:

(b) $f(x) = x^4 - 7x + 1,\ g(x) = 2x^2 + 1$ in $\mathbb{Q}[x]$.

(d) $f(x) = 4x^4 + 2x^3 + 6x^2 + 4x + 5,\ g(x) = 3x^2 + 1$ in $\mathbb{Z}_7[x]$.

6. Assume that R is a ring for which there exist $a, b \in R$ with $a \cdot b \neq 0_R$. Which of the following subsets of $R[x]$ are subrings of $R[x]$. Justify your answer:

(a) All polynomials with constant term 0_R.

(c) All polynomials of degree $\leq k$, where k is a fixed positive integer.

(d) All polynomials in which the odd powers of x, have zero coefficients.

(e) All polynomials in which the even powers of x, have zero coefficients.

14.

(a) Let R be an integral domain and $f(x),\ g(x) \in R[x]$. Assume that the leading coefficient of $g(x)$ is a unit in R. Verify that the Division Algorithm holds for $f(x)$ as dividend and $g(x)$ as divisor. [*Hint*: Adapt the proof of Theorem 4.6. Where is the hypothesis that F is a field used there?]

(b) Give an example in $\mathbb{Z}[x]$ to show that part (a) may be false if the leading coefficient of $g(x)$ is a not unit. [*Hint*: Exercise 5(b) with $\mathbb{Z}$ in place of $\mathbb{Q}$.]

16. Let R be a commutaive ring with identity and $a \in R$. If $1_R + ax$ is a unit in $R[x]$, show that $a^n = 0_R$ for some $n > 0$. [*Hint*: Suppose that the inverse of $1_R + ax$ is $b_0 + b_1x + b_2x^2 + \cdots + b_kx^k$. Since their product is 1_R, $b_0 = 1_R$ (Why?) and the other coefficients are all 0_R.]

Section 4.2

5. The Euclidean Algoritm for finding gcd is described for integers in Exercise 15 of Section 1.2. The process given there also works for polynomials over a field, with one minor adjustment. For integers the last nonzero remainder is the gcd. For polynomials the last nonzero remainder is a common divisor of highest degree, but it may not be monic. In that case, multiply it by the inverse of its leading coefficient to obtain the gcd. Use the Euclidean Algorithm to find the gcd of the following polynomials:

(b) $x^5 + x^4 + 2x^3 - x^2 - x - 2$ and $x^4 + 2x^3 + 5x^2 + 4x + 4$ in $\mathbb{Q}[x]$.

(d) $4x^4 + 2x^3 + 6x^2 + 4x + 5$ and $3x^3 + 5x^2 + 6x$ in $\mathbb{Z}_7[x]$.

14. Let $f(x), g(x), h(x) \in F[x]$ with $f(x)$ and $g(x)$ relatively prime. If $f(x)|h(x)$ and $g(x)|h(x)$, prove that $f(x)g(x)|h(x)$.