Math 301 Homework 5 (due on September 27, 2023)

Section 3.1

12. Let $\mathbb{Z}[i]$ denote the set $\{a+bi|a,\ b\in\mathbb{Z}\}$. Show that $\mathbb{Z}[i]$ is a subring of $\mathbb{C}$.

22. Define a new addition $\oplus$ and multiplication $\odot$ on $\mathbb{Z}$ by

$$a\oplus b=a+b-1 \text{ and } a\odot b=a+b-ab,$$

where the operation on the right-hand side of the equal signs are ordinary addition, subtraction, and multiplication. Prove that, with the new operations $\oplus$ and $\odot$, $\mathbb{Z}$ is an integral domain.

30. The addition table and part of the multiplication table for a four-element ring are given below. Use the distributive laws to complete the multiplication table.

$+$	w	x	y	z
w	w	x	y	z
x	x	y	z	w
y	y	z	w	x
z	z	w	x	y

and

$\cdot$	w	x	y	z
w	w	w	w	w
x	w	y		
y	w		w	
z	w		w	y

42. A **division ring** is a (not necessarily commutative) ring R with identity $1_R\neq 0_R$ that satisfies Axiom 11 and 12 (pages 48 and 49). Thus a field is a commutative division ring. See Exercise 43 for a noncommutative example. Suppose R is a division ring and a, b are nonzero elements of R.

(a) If $bb=b$, prove that $b=1_R$. [*Hint:* Let v be a solution of $bx=1_R$ and note that $bv=b^2v$.]

(b) If u is a solution of the equation $ax=1_R$, prove that u is also a solution of the equation $xa=1_R$. (Remember that R may not be commutative.) [*Hint:* Use part (a) with $b=ua$.]

Section 3.2

12. Let $a,\ b$ be elements of a ring R.

(a) Prove that the equation $a+x=b$ has a unique solution in R. (You must prove that there is a solution *and* that this solution is the only one.)

(b) If R is a ring with identity and a is a unit, prove that the equation $ax=b$ has a unique solution in R.

18. Let a be a nonzero element of a ring with identity. If the equation $ax = 1_R$ has a solution u and the equation $ya = 1_R$ has a solution v, prove that $u = v$.

22.

(a) If ab is a zero divisor in a ring R, prove that a or b is a zero divisor.

(b) If a or b is a zero divisor in a commutative ring R and $ab \neq 0$, prove that ab is a zero divisor.

33. Let R be a ring with identity. If ab and a are units in R. Prove that b is a unit.

40. An element a of a ring is **nilpotent** if $a^n = 0_R$ for some positive integer n. Prove that R has no nonzero nilpotent elements if and only if 0_R is the unique solution of the equation $x^2 = 0_R$.