Math 301 Homework 3 (due on September 13, 2023)

Section 2.1

6. If $a \equiv b \pmod{n}$ and $k|n$, is it true that $a \equiv b \pmod{k}$? Justify your answer.

12. If $p \geq 5$ and p is prime, prove that $[p] = [1]$ or $[p] = [5]$ in $\mathbb{Z}_6$. [Hint:Theorem 2.3 and Corollary 2.5.]

14.

(a) Prove or disprove: If $ab \equiv 0 \pmod{n}$, then $a \equiv 0 \pmod{n}$ or $b \equiv 0 \pmod{n}$

(b) Do part (a) when n is prime.

16. If $[a] = [1]$ in $\mathbb{Z}_n$, prove that $(a, n) = 1$. Show by example that the converse may be false.

20.

(a) Prove or disprove: If $a^2 \equiv b^2 \pmod{n}$, then $a \equiv b \pmod{n}$ or $a \equiv -b \pmod{n}$.

(b) Do part (a) when n is prime.

22.

(a) Give an example to show that the following statement is false: If $ab \equiv ac \pmod{n}$ and $a \not\equiv 0 \pmod{n}$, then $b \equiv c \pmod{n}$.

(b) Prove that the statement in part (a) is true whenever $(a, n) = 1$.

Appendix C

15. What is wrong with the following "proof" that all roses are the same color?

It suffices to prove the statement: In every set of n roses, all the roses in the set are the same color. If $n = 1$, the statement is certainly true. Assume the statement is true for $n = k$. Let S be a set of $k + 1$ roses. Remove one rose (call it rose A) from S; there are k roses remaining, and they must all be the same color by the induction hypothesis. Replace rose A and remove a different rose (call it rose B). Once again there are k roses remaining that must all be the same color by the induction hypothesis. Since the remaining roses include rose A, all the roses in S have the same color. This proves that the statement is true when $n = k + 1$. Therefore, the statement is true for all n by induction.

17. Let x be a real number greater than -1. Prove that for every positive integer n, $(1+x)^n \geq 1+nx$.

Appendix D

11. Let $\sim$ be defined on the set $\mathbb{R}^*$ of nonzero real numbers by $a \sim b$ if and only if $a/b \in \mathbb{Q}$. Prove that $\sim$ is an equivalence relation.

17. Let $\sim$ be a symmetric and transitive relation on a set A. What is wrong with the following "proof" that $\sim$ is reflexive: $a \sim b$ implies $b \sim a$ by symmetry; then $a \sim b$ and $b \sim a$ implies $a \sim a$ by transitivity. [Also see Exercise 8(f).]