Math 301 Homework 10 (due on November 15, 2023)

Section 7.1

4. Determine whether the set G is a group under the operation $*$.

(a) $G = \{2, 4, 6, 8\}$ in $\mathbb{Z}_{10}$; $a * b = ab$

(b) $G = \mathbb{Z}$; $a * b = a - b$

(c) $G = \{n \in \mathbb{Z} | n \text{ is odd }\}$; $a * b = a + b$

(d) $G = \{2^x | x \in \mathbb{Q}\}$; $a * b = ab$

8. Use Theorem 2.10 to list the elements of each of these groups: U_4, U_6, U_{10}, U_{20}, U_{30}.

10. Show that $G = \left\{ \left(\begin{array}{cc} a & b \\ -b & a \end{array} \right) \middle| a, b \in \mathbb{R}, \text{ not both } 0 \right\}$ is an abelian group under matrix multiplication.

24. Prove that the set of nonzero real numbers is a group under the operation $*$ defined by

$$a * b = \begin{cases} ab & \text{if } a > 0 \\ a/b & \text{if } a < 0. \end{cases}$$

30. A partial operation table for a group $G = \{e, a, b, c, d, f\}$ is shown below. Complete the table.

	e	a	b	c	d	f
e	e	a	b	c	d	f
a	a	b	e	d		
b	b					
c	c	f				a
d	d					
f	f					

Section 7.2

7. Find the order of the given element.

(b) $\left(\begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 3 & 7 & 5 & 1 & 4 & 6 \end{array} \right)$ in S_7.

(d) $\begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ -\frac{3}{2} & -\frac{1}{2} \end{pmatrix}$ in $GL(2,\mathbb{R})$.

12. If $a,\ b \in G$ and n is any integer. Show that $(aba^{-1})^n = ab^n a^{-1}$.

20.

(a) Show that $a = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$ has order 3 in $GL(2,\mathbb{R})$ and $b = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ has order 4.

(b) Show that ab has infinite order.

24. If $(ab)^2 = a^2b^2$ for all $a,\ b \in G$, prove that G is abelian.

34. Suppose G has order 4, but no element of order 4.

(a) Prove that no element of G has order 3. [*Hint:* If $|g| = 3$, then G consists of four distinct elements $g, g^2, g^3 = e, d$ Now gd must be one of these four elements. SHow that each possibility leads to contradiction.]

(b) Explain why every nonidentity element of G has order 2.

(c) Denote the elements of G by $e,\ a,\ b,\ c$ and write out the operation table for G.