Math 504 — Abstract Algebra I — Fall 2022 — Review Problems for Final Exam

By "ring" we mean "ring with 1". Ring homomorphisms and subrings are assumed to be unital. The dihedral group of order n is denoted D_n.

1. Find the group of units of $\mathbb{Z}[(1+\sqrt{-3})/2]$.
2. Describe all group homomorphisms $D_{16} \to Q_8$.
3. Find the smallest positive integer n such that the alternating group A_n contains an element σ of order 42. Then find the order of the centralizer of σ in A_n.
4. (Hard!) Let $G = \langle x, y, z \mid x^2 = y^3 = z,\ z^2 = 1,\ zx = xz,\ zy = yz \rangle$. Show there is an isomorphism $G \to \mathrm{SL}(2,\mathbb{Z})$. (*Hint:* Consider the matrices $X = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, $Y = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$, $Z = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$.)
5. Let $G = \langle u, v, w \mid uv = vu,\ vw = wv,\ wu = uw,\ (uvw)^4 = 1,\ u^4v^6w^8 = 1 \rangle$. Describe the torsion subgroup of G.
6. A Christmas decoration in the shape of an (irregular) octahedron with vertices $(\pm 1, 0, 0)$, $(0, \pm 1, 0)$, $(0, 0, \pm 3)$ is to be painted using three colors (red, green, gold). How many possibilities are there? (The decoration can be turned up-side-down. Each side has one color and not all colors need to be used.)
7. Let G be a group of order 315 having a normal Sylow 3-subgroup. Prove that $Z(G)$ contains a Sylow 3-subgroup and deduce that G is abelian.
8. (a) Show that no group of order $3^4 \cdot 5 \cdot 11$ is simple.

   (b) Show that no group of order 768 is simple.
9. Use semidirect products to construct a nonabelian group of order $p^2(p^2 - p)/4$ for each prime p congruent to 1 mod 4.
10. Let R be a (not necessarily commutative) ring. Let P be an ideal of R. Prove that the following two statements are equivalent.

    (a) If $P \supseteq IJ$ for some ideals I, J of R, then $P \supseteq I$ or $P \supseteq J$.

    (b) If $aRb \subseteq P$ for some elements $a, b \in R$, then $a \in P$ or $b \in P$.
11. Let R be the ring $\begin{bmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} \end{bmatrix}$. Show that every maximal ideal is of the form $\begin{bmatrix} p\mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} \end{bmatrix}$ or $\begin{bmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & p\mathbb{Z} \end{bmatrix}$ where p is a prime in $\mathbb{Z}$. (Note: R is not commutative!) (*Hint:* First prove that any maximal ideal must contain $\begin{bmatrix} 0 & \mathbb{Z} \\ 0 & 0 \end{bmatrix}$. Then use the second isomorphism theorem and lattice isomorphism theorem for rings.)
12. Let R be a commutative ring of characteristic zero, and let P be a prime ideal of R. Prove that if P is finite then R/P has characteristic zero. Does the converse hold?

13. Let $\mathbb{F}$ be a field and let $f(x) \in \mathbb{F}[x]$ be a nonzero polynomial. Let $R = \mathbb{F}[x,y]/(f(x))$. Prove that R is an integral domain if and only if R is a Euclidean domain.

14. Which of the following statements are true?

    (i) Every integral domain is isomorphic to a subring of a Euclidean domain.

    (ii) If R is a PID then $R[x]$ is a PID.

    (iii) Every commutative ring is isomorphic to a subring of a field.

    (iv) If R is a Euclidean domain then $R[x]$ is a Euclidean domain.

    (v) If R is a UFD then R/P is a PID for every prime ideal P of R.

    (vi) Every ring of characteristic zero is commutative.

15. Find all solutions to the system of congruence equations in $\mathbb{Z}[i]$:

$$\begin{cases} x \equiv 1 + i \pmod{2 + 5i} \\ x \equiv 2 - i \pmod{3 - 2i} \end{cases}$$

16. Let $\mathbb{F}_4 = \mathbb{Z}_2[x]/(x^2 + x + 1)$. Find all irreducible polynomials $f(y)$ of degree at most three in $\mathbb{F}_4[y]$.