Math 504 — Abstract Algebra I — Fall 2022 — Review Problems for Final Exam (Selected Solutions)

4. Let $G = \langle x, y, z \mid x^2 = y^3 = z,\ z^2 = 1,\ zx = xz,\ zy = yz\rangle$. Show there is an isomorphism $G \to \mathrm{SL}(2,\mathbb{Z})$. (*Hint:* Consider the matrices $X = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, $Y = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$, $Z = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$.)

   Solution: Define $\phi : \{x, y, z\} \to \mathrm{SL}(2,\mathbb{Z})$ by $\phi(x) = X,\ \phi(y) = Y,\ \phi(z) = Z$. By the universal property of free groups, f extends to a group homomorphism $\Phi : F(x, y, z) \to \mathrm{SL}(2,\mathbb{Z})$ such that $\Phi(x) = X,\ \Phi(y) = Y,\ \Phi(z) = Z$. By definition, $G = F(x, y, z)/N$ where N is the normal subgroup of $F(x, y, z)$ generated by the five elements x^2z^{-1}, y^3z^{-1}, z^2, $zxz^{-1}x^{-1}$, $zyz^{-1}z^{-1}$. One checks that each of these is contained in the kernel of Φ. For example,

   $$\Phi(x^2z^{-1}) = \Phi(x)^2\Phi(z)^{-1} = X^2Z^{-1} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = I.$$

   Therefore, Φ induces a well-defined group homomorphism

   $$\varphi : G \to \mathrm{SL}(2,\mathbb{Z})$$

   satisfying $\varphi(x) = X$, $\varphi(x) = Y$, $\varphi(z) = Z$.

   Next, we show that φ is surjective. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}(2,\mathbb{Z})$. We need to show $A \in \varphi(G)$, meaning A can be written as a product of the matrices X, Y, Z in some order. We proceed by induction on $N = \min\{|a|, |c|\}$. For the base case, suppose $N = 0$. Then, since $XA = \begin{bmatrix} -c & -d \\ a & b \end{bmatrix}$, we may without loss of generality assume that $|c| = N = 0$. Since $ad - bc = 1$, we get $(a, d) \in \{(1, 1), (-1, -1)\}$. After multiplying by Z we may assume $(a, d) = (1, 1)$. Then $A = (XYZ)^b$, since $XYZ = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. For the induction step, suppose $N > 0$. Then $XA = \begin{bmatrix} -c & -d \\ a & b \end{bmatrix}$ shows we may assume $ac > 0$. By multiplying by Z we may assume $a > 0$ and $c > 0$. If $a > c$, note that

   $$Y^2XZ\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a - c & b - d \\ c & d \end{bmatrix} \in \varphi(G)$$

   by the induction hypothesis, thus $A \in \varphi(G)$. Similarly, if $a < c$, then we get

   $$YXZ\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c - a & d - b \end{bmatrix} \in \varphi(G)$$

   by induction, hence $A \in \varphi(G)$. This proves the induction step. Therefore $A \in \varphi(G)$.

   Lastly, we only sketch the proof of injectivity of φ. First observe that by the relations in G, any element of G can be written uniquely in the form

   $$x^bz^c, \quad \text{or} \quad x^a(y^{1+j_1}xz)(y^{1+j_2}xz)\cdots(y^{1+j_{k-1}}xz)y^{1+j_k}x^bz^c$$

   where $k \in \mathbb{Z}_{\geq 1}, j_i \in \{0, 1\}, a, b, c \in \{0, 1\}$. Then prove that each of those expressions map to different matrices in $\mathrm{SL}(2,\mathbb{Z})$.

5. Let $G = \langle u, v, w \mid uv = vu,\ vw = wv,\ wu = uw,\ (uvw)^4 = 1,\ u^4v^6w^8 = 1\rangle$. Describe the torsion subgroup of G.

Solution: Since G is generated by u, v, w and those generators pairwise commute, G is abelian. By the universal property of free abelian groups, there is a group homomorphism $\Phi : \mathbb{Z}^3 \to G$ sending $e_1 \mapsto u, e_2 \mapsto v, e_3 \mapsto w$. Since u, v, w generate G, the homomorphism Φ is surjective. Due to the relations in G, the kernel of Φ is generated by $4(e_1 + e_2 + e_3) = \begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix}$ and $4e_1 + 6e_2 + 8e_3 = \begin{bmatrix} 4 \\ 6 \\ 8 \end{bmatrix}$. Bringing $A = \begin{bmatrix} 4 & 4 \\ 4 & 6 \\ 4 & 8 \end{bmatrix}$ to its Smith normal form we get $\begin{bmatrix} 2 & 0 \\ 0 & 4 \\ 0 & 0 \end{bmatrix}$. We may append a column of zeroes to make it have as many columns as there are generators in $\mathbb{Z}^3$. Thus

$$G \cong \mathbb{Z}^3/K \cong \mathbb{Z}^3/(2\mathbb{Z} \times 4\mathbb{Z} \times 0\mathbb{Z}) \cong \mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}.$$

The torsion subgroup of this is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}^4$.

9. Use semidirect products to construct a nonabelian group of order $p^2(p^2 - p)/4$ for each prime p congruent to 1 mod 4.

Solution: It suffices to construct a nonabelian group G of order $(p^2 - p)/4$ as then $\mathbb{Z}_{p^2} \times G$ is nonabelian of order $p^2(p^2 - p)/4$. For primes p we know that $\mathrm{Aut}(\mathbb{Z}_p)$ is cyclic of order $p - 1$. Let $g \in \mathrm{Aut}(\mathbb{Z}_p)$ be a generator. By assumption, 4 divides $p - 1$. Let $h = g^4$. Then h has order $d := (p - 1)/4$ which gives a group homomorphism $\alpha : \mathbb{Z}_d \to \mathrm{Aut}(\mathbb{Z}_p)$ determined by sending $[1]_d$ to g. Then the semidirect product $G = \mathbb{Z}_p \rtimes_\alpha \mathbb{Z}_d$ is a nonabelian group of order $pd = (p^2 - p)/4$.