Math 504 — Abstract Algebra I — Fall 2022 — Review Problems for Exam 2

1. How many abelian groups are there of order 24,069,811,311? (You don't need to list them.)
   Hint: $24{,}069{,}811{,}311 = 3^3 \cdot 7^4 \cdot 13^5$.

2. Classify (that is, list one from each isomorphism class) all abelian groups of order $7^4 \cdot 13^2$ up to isomorphism. Give their invariant factors and elementary divisors.

3. Let K be the subgroup of $\mathbb{Z}^n$ generated by $\{\mathbf{e}_i - \mathbf{e}_{i+1}\}_{i=1}^{n-1}$ where $\mathbf{e}_i = (0, \ldots, \overset{i}{1}, \ldots, 0)$ are the standard basis vectors for $\mathbb{R}^n$. Find a product of cyclic groups isomorphic to $\mathbb{Z}^n/K$.

4. Let G and H be finite groups whose orders are relatively prime. Show that $\mathrm{Aut}(G \times H) \cong \mathrm{Aut}(G) \times \mathrm{Aut}(H)$.

5. Let G be a finite group, p a prime number and k a non-negative integer. Prove that the number of subgroups of G of index p^k is congruent modulo p to the number of normal subgroups of G of index p^k.
   Hint: Let G act by conjugation on the set of subgroups of index p^k.

6. Let G be a simple group of order 360. Show that every subgroup of G has index at least 6.

7. Let $\sigma = (1\ 2\ 3\ 4)(5\ 6\ 7\ 8) \in A_8$. What is the size of the conjugacy class of σ in A_8?

8. How many conjugacy classes are there in S_6 consisting of elements of odd order?

9. Let G be a group of order $2907 = 9 \cdot 17 \cdot 19$. Prove that G cannot be a simple group.

10. Show that every group of order $18785 = 5 \cdot 13 \cdot 17^2$ is abelian.
    Hint: First show that the center has to contain large subgroups.

11. Let N and H be any two abelian groups, and $\alpha : H \to \mathrm{Aut}(N)$, $h \mapsto \alpha_h$ be a homomorphism. Show that $Z(N \rtimes_\alpha H) = N^H \times (\ker \alpha)$ where $N^H = \{x \in N \mid \alpha_h(x) = x\ \forall h \in H\}$.

12. Let N and H be any groups, and $\alpha : H \to \mathrm{Aut}(N)$, $h \mapsto \alpha_h$ be a homomorphism. Show that the external semidirect product $N \rtimes_\alpha H$ is abelian if and only if N and H are abelian, and $\alpha_h = \mathrm{Id}_N$ for all $h \in H$.

13. Show that for any set X, the group $\mathrm{Aut}\big(F(X)\big)$ contains a subgroup isomorphic to S_X.
    Hint: Use the universal property of free groups.