Math 504 — Abstract Algebra I — Fall 2022
Selected Solutions to Review Problems for Exam 2

5. Let G be a finite group, p a prime number and k a non-negative integer. Prove that the number of subgroups of G of index p^k is congruent modulo p to the number of normal subgroups of G of index p^k.
Hint: Let G act by conjugation on the set of subgroups of index p^k.

   Solution: Let X be the set of all subgroups of G of index p^k. Let G act on X by conjugation. Let X^G be the union of the singleton orbits. That is, $X^G = \{H \in X \mid gHg^{-1} = H\, \forall g \in G\} = \{H \in X \mid H \trianglelefteq G\}$. So X^G is the set of all normal subgroups of G of index p^k. Let $\mathcal{O}_1, \ldots, \mathcal{O}_r$ be the non-singleton orbits. By the Orbit Decomposition Theorem, we have

$$|X| = |X^G| + \sum_{i=1}^{r} |\mathcal{O}_i|$$

   We are done if we can show that p divides $|\mathcal{O}_i|$ for each i since then $|X| \equiv |X^G| \pmod p$. By the Orbit-Stabilizer Theorem, $|\mathcal{O}_i| = |G : \mathrm{Stab}_G(H_i)| = |G : N_G(H_i)|$ for any choice of $H_i \in \mathcal{O}_i$. Since $\mathcal{O}_i$ is a non-singleton orbit, $|G : N_G(H_i)| > 1$. On the other hand $H_i \leq N_G(H_i)$ so that $|G : N_G(H_i)| = \frac{|G:H|}{|N_G(H_i):H|}$ which is a power of p, since H has index p^k. Thus p divides $|G : N_G(H_i)|$ for each i.

6. Let G be a simple group of order 360. Show that every **nontrivial** subgroup of G has index at least 6.

   Solution: Let H be a nontrivial subgroup of G. Let $k = |G : H|$. Seeking a contradiction, assume that $k \leq 5$. Let G act by left multiplication on G/H. That is, $g.(g'H) = (gg')H$ for all $g \in G$ and $g'H \in G/H$. The permutation representation afforded by this action is a homomorphism $\pi : G \to S_{G/H} \cong S_k$. Since G is simple, $\ker \pi$ is either 1 or G. For $g \in G$, $g \notin H$, we have $g.H = gH \neq H$ so the action is not trivial, therefore $\ker \pi \neq G$. So $\ker \pi = 1$. But that means π is injective from a set with 360 elements to a set with $k! \leq 5! = 120$ elements which is a contradiction.

8. How many conjugacy classes are there in S_6 consisting of elements of odd order?

   Solution: Conjugacy classes in S_6 correspond to partitions of 6 via cycle types. The order of an element is the lcm of its cycle lengths. So the question becomes: How many partitions are there of 6 into a sum of odd parts? The only possibilities are 5+1, 3+3, 3+1+1+1, 1+1+1+1+1+1. The corresponding conjugacy classes have representatives $(12345), (123)(456), (123), (1)$. The answer is 4.

13. Show that for any set X, the group $\mathrm{Aut}\big(F(X)\big)$ contains a subgroup isomorphic to S_X.
Hint: Use the universal property of free groups.

   Solution: Let $\sigma \in S_X$ and define a function $f_\sigma : X \to F(X)$ by $f_\sigma(x) = \sigma(x)$ for all $x \in X$. By the universal property of free groups, f_σ extends uniquely to a group homomorphism $\varphi_\sigma : F(X) \to F(X)$. If $\sigma, \tau \in S_X$ then for all $x \in X$ we have $(\varphi_\sigma \circ \varphi_\tau)(x) = \varphi_\sigma\big(\varphi_\tau(x)\big) = \varphi_\sigma(\tau(x)) = \sigma(\tau(x)) = (\sigma\tau)(x)$. But also, $\varphi_{\sigma\tau}(x) = (\sigma\tau)(x)$ for all $x \in X$. So by the uniqueness part of the universal property of free groups,

$$\varphi_{\sigma\tau} = \varphi_\sigma \circ \varphi_\tau, \qquad \forall \sigma, \tau \in S_X. \tag{0.1}$$

Taking $\tau = \sigma^{-1}$ and using that $\varphi_{\mathrm{Id}_X} = \mathrm{Id}_{F(X)}$ we obtain that $\varphi_{\sigma^{-1}} \circ \varphi_\sigma = \mathrm{Id}_{F(X)}$. Changing σ to σ^{-1}, $\varphi_\sigma \circ \varphi_{\sigma^{-1}} = \mathrm{Id}_{F(X)}$. Thus φ_σ is invertible. So $\varphi_\sigma \in \mathrm{Aut}(F(X))$. This defines a function

$$\psi : S_X \to \mathrm{Aut}(F(X)), \quad \psi(\sigma) = \varphi_\sigma.$$

Equation (0.1) shows ψ is a homomorphism. We claim ψ is injective. Suppose $\psi(\sigma) = \psi(\tau)$. That is, $\varphi_\sigma = \varphi_\tau$. Applying both sides to an arbitrary $x \in X$ we get $\sigma(x) = \tau(x)$. Since x was arbitrary, $\sigma = \tau$. Therefore $S_X \cong \psi(S_X) \leq \mathrm{Aut}(F(X))$.