Math 504 — Abstract Algebra I — Fall 2022 — Review Problems for Exam 1

1. What is the order of the group $(\mathbb{Z}/1000\mathbb{Z})^\times$? Is there a positive integer n such that $(\mathbb{Z}/n\mathbb{Z})^\times$ has order 55? What about 56?

2. Let $D_{2\infty} = \mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z})$. We introduce a binary operation $*$ on $D_{2\infty}$ by
$$(a, \bar{b}) * (c, \bar{d}) = \big(a + (-1)^b c,\ \bar{b} + \bar{d}\big), \qquad \forall a, b \in \mathbb{Z}$$
where $\bar{a} = [a]_2$ denotes the congruence class of a modulo 2. Show that the operation is well-defined and that $\big(D_{2\infty}, *\big)$ is a group and that it is nonabelian.

3. Find the group of invertible elements in the monoid $\mathbb{Z}^n$ under multiplication.

4. Find the cycle decomposition and order of $\sigma\tau$ and $\tau\sigma$ if
$$\sigma = (5\ 2\ 6\ 8\ 4)(3\ 5\ 7)(1\ 5\ 2), \qquad \tau = (3\ 7\ 1\ 6\ 2)(7\ 2\ 5\ 6)$$

5. Show that the set of complex $n \times n$-matrices whose determinant has absolute value 1 is a normal subgroup of $\mathrm{GL}(n, \mathbb{C})$.

6. Suppose that G acts on a set X. Fix $x \in X$ and let $H = \mathrm{Stab}_G(x)$. Show that $\varphi : G/H \to X$, $\varphi(gH) = g.x$ is a well-defined and injective function.

7. Let G be a group acting on a set X. Show that G acts on X^X (the set of all functions from X to X) by
$$(g.\varphi)(x) = g.\big(\varphi(g^{-1}.x)\big) \qquad \forall g \in G, \varphi \in X^X, x \in X.$$

8. Let G and H be two groups. Prove that $Z(G \times H) = Z(G) \times Z(H)$.

9. (a) State the definition of the centralizer $C_G(a)$ of an element a in a group G.

   (b) Find the centralizer of s in the dihedral group of order 16.

10. Show that $\mathrm{SL}_{p-1}(\mathbb{Z}/p\mathbb{Z})$ has non-trivial center for any prime p.

11. Prove the following non-isomorphisms of groups:

    (a) $S_8 \not\cong Q_8$.

    (b) $\mathbb{Z} \not\cong \mathbb{Z} \times \mathbb{Z}$

    (c) $Q_8 \times Q_8 \not\cong D_{16}$.

12. If $A, B \subseteq G$ and $AB = BA$, show that $\langle A \rangle \langle B \rangle = \langle B \rangle \langle A \rangle$.

13. Let G be a group and $g, h \in G$. Prove that h and ghg^{-1} have the same order.

14. List all elements of A_4 of even order.

15. Prove that any quotient group of a cyclic group is cyclic.

16. Suppose $\varphi : G_1 \to G_2$ is a surjective homomorphism between finite groups. Show that $|G_2|$ divides $|G_1|$.

17. Show that if $N_1 \trianglelefteq G_1$ and $N_2 \trianglelefteq G_2$ then

$$(G_1 \times G_2)/(N_1 \times N_2) \simeq (G_1/N_1) \times (G_2/N_2).$$

18. (a) State the Second Isomorphism Theorem.

    (b) Suppose that $H \leq G$, $N \trianglelefteq G$ and $\gcd(|H|, |N|) = 1$. Prove that $HN/N \simeq H$.