Deformation theory

(Hochschild, Gerstenhaber)

A algebra $/\mathbb{k}$, $\hbar$ indeterminate

$\mathbb{k}[\hbar]$, $\mathbb{k}[[\hbar]]$, $\mathbb{k}[\hbar]\otimes_{\mathbb{k}} A$, $\underbrace{\mathbb{k}[[\hbar]]\otimes_{\mathbb{k}} A}$

$A[[\hbar]] = \left\{ \sum_{n=0}^{\infty} a_n \hbar^n \,/\, a_n \in A \right\}$.

largest

$\left\{ \sum a_n \otimes \hbar^n \,\middle|\, a_n \in A \text{ \& } \dim \operatorname{Span}\{a_n\} < \infty \right\}$

$f_1 \otimes b_1 + \dots + f_n \otimes b_n$

$= \sum_{k=0}^{\infty} a_k \otimes \hbar^n \implies a_k \in \operatorname{Span}\{b_1, \dots, b_n\}$

New multiplication in $A[[\hbar]]$:

$$\mu : A[[\hbar]] \underset{k[[\hbar]]}{\otimes} A[[\hbar]] \longrightarrow A[[\hbar]]$$

- $\mu(a \otimes b) = \underbrace{\mu_0(a\otimes b)}_{\text{assume } = ab} + \mu_1(a\otimes b)\hbar + \dots$

  $a, b \in A$

- μ $k[[\hbar]]$-linear
- μ continuous wrt. $(\hbar)$-adic topology
- $\mu(1 \otimes b) = b = \mu(b \otimes 1)$ $\forall b \in A$.

Suppose we have such μ. When is $(A[[\hbar]], \mu, 1)$ an associative algebra?

<u>Def</u> Such an alg is a (formal) **deformation** of A.

<u>Def</u> An **infinitesimal deformation** (a.k.a. 1st order deformation) of A is a triple $(A[[\hbar]], \mu, 1)$ such that μ is associative mod $\hbar^2$.

$$\mu(a \otimes \mu(b \otimes c)) - \mu(\mu(a \otimes b) \otimes c) =$$

$$= \mu(a \otimes (bc + \mu_1(b \otimes c)\hbar + \dots))$$

$$- \mu((ab + \mu_1(a \otimes b)\hbar + \dots) \otimes c) =$$

$$= a\left(\cancel{bc} + \mu_1(b\otimes c)\hbar + \cdots\right)$$

$$+ \mu_1\left(a \otimes \left(bc + \mu_1(b\otimes c)\hbar + \cdots\right)\right)\hbar + \cdots$$

$$- \left[\left(\cancel{ab} + \mu_1(a\otimes b)\hbar + \cdots\right)c + \mu_1\left(\left(ab + \mu_1(a\otimes b)\hbar + \cdots\right)\otimes c\right)\hbar + \cdots\right]$$

$$= \overbrace{\left[a\,\mu_1(b\otimes c) + \mu_1(a\otimes bc) - \mu_1(a\otimes b)c - \mu_1(ab\otimes c)\right]}^{(*)}\hbar + \hbar^2(\cdots)$$

<u>So</u> $(A[[\hbar]], \mu, 1)$ is an infinitesimal deformation of A iff $(*) = 0$ $\forall a, b, c \in A$.

$(\ast) = 0 \iff$

$$a\mu_1(b\otimes c) - \mu_1(ab\otimes c) + \mu_1(a\otimes bc) - \mu_1(a\otimes b)c = 0$$

<u>Def</u> A Hochschild n-cocycle on A is a linear map $f: A^{\otimes n} \to A$ such that

$$(\ast\ast)\quad \begin{cases} a_0 f(a_1\otimes\cdots\otimes a_n) + \sum_{k=1}^{n} (-1)^k f(a_0\otimes\cdots\otimes \underbrace{a_{k-1}a_k}_{k\text{:th slot}}\otimes\cdots\otimes a_n) + \\ \qquad + (-1)^{n+1} f(a_0\otimes a_1\otimes\cdots\otimes a_{n-1})a_n = 0 \end{cases}$$

$\forall\, a_0, a_1, \ldots, a_n \in A.$

Hochschild Cohomology.

<u>Def</u> An n-cochain on A is a linear map $f : A^{\otimes n} \to A$. Let $C^n(A)$ be the set of n-cochains on A

<u>Def</u> Define the coboundary operator $d = d^n : C^n(A) \to C^{n+1}(A)$ by $(df)(a_0 \otimes a_1 \otimes \cdots \otimes a_n) = \text{LHS of } (**)$ $\forall f \in C^n(A)$.

Problem:

Lemma $\boxed{d \circ d = 0}$ $\quad (d^{n+1} \circ d^n = 0 \ \forall n)$

Def $Z^n(A) := \ker d^n$ = set of n-cocycles

Def $B^n(A) := \operatorname{im} d^{n-1}$ = set of n-coboundaries

Note $B^n(A) \subset Z^n(A)$ since $d^2 = 0$

Def $H^n(A) := Z^n(A) / B^n(A)$

= n^{th} Hochschild cohomology group

Let $f \in C^1(A) = \mathrm{End}(A)$

$$(df)(a_0 \otimes a_1) = a_0 f(a_1) - f(a_0 a_1) + f(a_0) a_1$$

So $Z^1(A) = \{ f : A \to A \mid f \text{ is a derivation} \}$

$$= \mathrm{Der}(A)$$

Let $f \in C^0(A) = \mathrm{Hom}_{\mathbb{k}}(\underbrace{A^{\otimes 0}}_{\mathbb{k}}, A^{\otimes 1}) \cong A$

$1 \mapsto a$

Write $f = b \in A$

$$(\partial b)(a_0) = b a_0 - a_0 b = [b, a_0]$$

$\Rightarrow \partial b$ is an inner derivation.

$$\Rightarrow \quad H^1(A) = \frac{\mathrm{Der}(A)}{\mathrm{Inn}(A)} =: \mathrm{Out}(A)$$

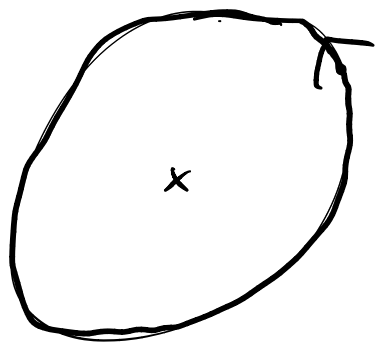
subspace of inner derivations

<u>Remarks</u>

1) Infinitesimal deformations of A are "equivalent" iff the μ_i's differ by a 2-coboundary.

$$\Rightarrow \{\text{Infinitesimal deformations of } A\}/\sim \;\cong\; H^2(A).$$

2) If $H^3(A) = 0$ then any infinitesimal deformation can be extended to a deformation, and conversely.



$A = \mathbb{C}[z, z^{-1}]$

$H^1(A) \neq 0$?