Limits.

$$A = \lim_{j} A_j$$

$\mathcal{J}$ small category

$F : \mathcal{J} \longrightarrow \mathcal{C}$ functor: $\forall\, i \to j$ in $\mathcal{J}$: $A_i \longrightarrow A_j$

$j \longmapsto A_j$

$A \in \mathcal{C}$ with $\pi_j : A \longrightarrow A_j$ s.t.

$\forall\, i \to j$ in $\mathcal{J}$:

$$\begin{tikzcd}
 & A \arrow[dl, "\pi_i"'] \arrow[dr, "\pi_j"] \arrow[d, phantom, "2"] & & B \arrow[ll, dashed] \arrow[dlll] \arrow[dl] \\
A_i \arrow[rr] & {} & A_j &
\end{tikzcd}$$

Ex 1 $J = \{1, 2, 3, \dots\}$ $\quad i \xrightarrow{!} j$ iff $j \mid i$

$F : j \longmapsto \mathbb{Z}/j\mathbb{Z} \in \underline{Ab} = \mathcal{C}$

$i \to j \implies \mathbb{Z}/i\mathbb{Z} \twoheadrightarrow \mathbb{Z}/j\mathbb{Z}$

$j \mid i \qquad\qquad [a]_i \longmapsto [a]_j$

$\Leftrightarrow i\mathbb{Z} \subset j\mathbb{Z}$

$(6\mathbb{Z} \subset 3\mathbb{Z})$

"projective limit of finite things"

Def The profinite integers is

$$\hat{\mathbb{Z}} = \varprojlim \mathbb{Z}/n\mathbb{Z} =$$

$$= \lim \left(\begin{tikzcd} & \mathbb{Z}/2\mathbb{Z} \arrow[dl] & \mathbb{Z}/4\mathbb{Z} \arrow[l] & \cdots \\ \mathbb{Z}/1\mathbb{Z} & \mathbb{Z}/3\mathbb{Z} \arrow[l] & \mathbb{Z}/6\mathbb{Z} \arrow[ul] \arrow[l] & \cdots \\ & & \mathbb{Z}/9\mathbb{Z} \arrow[ul] & \cdots \\ & \mathbb{Z}/5\mathbb{Z} \arrow[uul] & \vdots & \ddots \\ & \mathbb{Z}/p\mathbb{Z} \arrow[uuul] & & \ddots \\ \vdots & \vdots & & \end{tikzcd} \right)$$

More concretely:

$$\hat{\mathbb{Z}} = \left\{ (a_n)_{n=1}^{\infty} \in \prod_{n=1}^{\infty} \mathbb{Z}/n\mathbb{Z} \;\middle|\; j \mid i \Rightarrow a_i \equiv a_j \pmod{j} \right\}$$

$$\mathbb{Z} \hookrightarrow \hat{\mathbb{Z}}, \quad a \longmapsto ([a]_n)_{n=1}^{\infty}$$

$$\pi_n : \hat{\mathbb{Z}} \to \mathbb{Z}/n\mathbb{Z}$$
$$(a_k)_{k=1}^{\infty} \mapsto a_n$$

$$\underset{\text{n=1}}{}\ \ \underline{\text{Ex.}}\quad 17 \longmapsto (\overset{n=1}{0}, \overset{2}{1}, \overset{3}{2}, \overset{4}{1}, \overset{5}{2}, \overset{6}{5}, 3, \ldots$$

$$\ldots, \overset{n=16}{1}, \overset{17}{0}, \overset{18}{17}, \overset{19}{17}, 17, \ldots)$$

Consider $\hat{a} = (a_n)_{n=1}^{\infty}$, $a_n = \begin{cases} 1 & , p \mid n \\ 0 & , p \nmid n \end{cases}$
(p fixed)

if $j \mid i$ is $a_i \equiv a_j \pmod{j}$?

<u>Problem</u>: Find $\hat{a} \in \hat{\mathbb{Z}} \setminus \mathbb{Z}$.

Ex ② $\mathbb{k}[\![t]\!] \cong \varprojlim \frac{\mathbb{k}[t]}{(t)^n} =$

$$= \lim \left(\frac{\mathbb{k}[t]}{(t)} \longleftarrow \frac{\mathbb{k}[t]}{(t)^2} \longleftarrow \frac{\mathbb{k}[t]}{(t)^3} \longleftarrow \cdots \right)$$

$$\pi_n : \mathbb{k}[\![t]\!] \longrightarrow \frac{\mathbb{k}[t]}{(t)^n}, \quad \sum_{k=0}^{\infty} c_k t^k \mapsto \sum_{k=0}^{n-1} c_k t^k$$

Ex ③ R ring, $I \subset R$ ideal.

$$R_I := \varprojlim R/I^n = \lim \left(R/I \leftarrow R/I^2 \leftarrow \cdots \right)$$

$\left.\begin{array}{l}\text{in } \underline{\mathrm{Ab}} \\ R_I \text{ has a ring structure.}\end{array}\right\}$ **OR**: in $\underline{\mathrm{Rng}}$

Topology. X set

Def A **topology** on X is a subset $\tau \subset \mathcal{P}(X)$ ← power set of X

such that

1) $\emptyset, X \in \tau$
2) if $U_\alpha \in \tau$, $\alpha \in I \Rightarrow \bigcup_{\alpha \in I} U_\alpha \in \tau$
3) if $U_1, \dots, U_n \in \tau \Rightarrow U_1 \cap \dots \cap U_n \in \tau$

Elements of τ are called **open** (wrt. τ).

A topological space is a set X together with a topology τ on X.

A map $f: X \to Y$ (X, Y top. spaces) is continuous if

$$\forall U \in \tau_Y : f^{-1}(U) \in \tau_X.$$

Ex. The discrete topology on a set X is $\tau = \mathcal{P}(X)$. Equivalently, it's the topology s.t. $\{x\} \in \tau \; \forall x \in X$.

Def If $\mathcal{A} \subset \mathcal{P}(X)$, the topology generated by $\mathcal{A}$, $\langle \mathcal{A} \rangle$, is the unique smallest topology containing $\mathcal{A}$:

$$\langle \mathcal{A} \rangle = \{\text{arbitrary unions of finite intersections of sets in } \mathcal{A}\}$$

The limit topology on

$$A = \lim A_j \quad \text{in } \mathcal{C} = \text{concrete category}$$

i.e. $\exists\, \mathcal{C} \to \underline{Set}$

is the topology on A with fewest open sets such that $\pi_j : A \to A_j$ are all continuous, where the A_j's are given the discrete topology.

That is it's the topolgy generated by $\mathcal{A} = \left\{ \pi_j^{-1}(\{x\}) \;\middle|\; \begin{array}{l} j \in J \\ x \in A_j \end{array} \right\}$

Say $\mathcal{C} = \underline{Ab}$. (Then $\ker \pi_j$ are open for all j) $\ker \pi_j = \pi_j^{-1}(\{0\})$.

<u>Note</u> A map $f: X \to Y$, Y discrete is cont. iff f is locally constant (Exercise). i.e. $\forall x \in X \; \exists \, U \in \mathcal{T}_X, x \in U : f|_U$ is constant

$A = \lim A_j$ w/ limit top.

$\pi_j : A \to A_j$ continuous.

discrete

Then $f : A \to B$ is continuous

iff f factors through some π_j:

$\exists n$:

$$
\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dr, "\pi_n"'] & & B \\
& A_n \arrow[ur, dashed] &
\end{tikzcd}
$$

__Example__ A continuous group homorphism

$$\hat{\mathbb{Z}} \longrightarrow \mathbb{k}^{\times} \quad \left(= GL(1,\mathbb{k})\right)$$

$\Longleftrightarrow$ a group hom. ~~map~~ $\mathbb{Z}/n\mathbb{Z} \to \mathbb{k}^{\times}$ for some n

$$\left\{ \begin{array}{c} \text{Roots of} \\ \text{unity} \end{array} \right\} \longleftrightarrow \text{1-dim'l } \underline{\text{continuous}} \text{ reps of } \hat{\mathbb{Z}}$$

$$\begin{array}{ccc} \varepsilon & 1 \longmapsto & \hat{\mathbb{Z}} \to \mathbb{Z}/n\mathbb{Z} \longrightarrow k^{\times} \\ \varepsilon^n = 1 & & [1] \longmapsto \varepsilon \end{array}$$