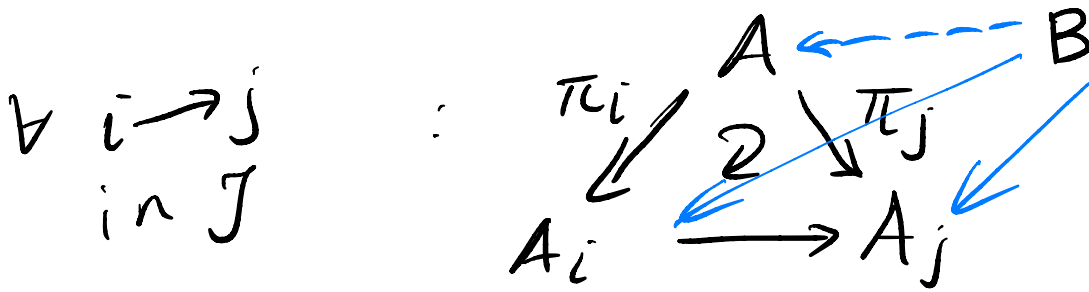


Limits. $A = \lim_j A_j$

\mathcal{J} small category

$F: \mathcal{J} \rightarrow \mathcal{C}$ functor: $\forall i \rightarrow j$ in \mathcal{J} : $A_i \rightarrow A_j$
 $j \mapsto A_j$

$A \in \mathcal{C}$ with $\pi_j: A \rightarrow A_j$ s.t.



Ex. ① $J = \{1, 2, 3, \dots\}$ $i \xrightarrow{!} j$ iff $j|i$

$F: j \mapsto \mathbb{Z}/j\mathbb{Z} \in \underline{Ab} = \mathcal{C}$

$i \rightarrow j \Rightarrow \mathbb{Z}/i\mathbb{Z} \rightarrow \mathbb{Z}/j\mathbb{Z}$

$j|i$

$[a]_i \mapsto [a]_j$

$\Leftrightarrow i\mathbb{Z} \subset j\mathbb{Z}$

$(6\mathbb{Z} \subset 3\mathbb{Z})$ "projective limit of finite things"

Def The profinite integers is

$$\hat{\mathbb{Z}} = \varprojlim \mathbb{Z}/n\mathbb{Z} =$$

$$= \lim \left(\begin{array}{cccc} & & \mathbb{Z}/2\mathbb{Z} & \leftarrow & \mathbb{Z}/4\mathbb{Z} & \cdots & \cdots & \cdots \\ & & \swarrow & & \swarrow & & & \\ \mathbb{Z}/1\mathbb{Z} & \leftarrow & \mathbb{Z}/3\mathbb{Z} & \leftarrow & \mathbb{Z}/6\mathbb{Z} & \cdots & \cdots & \\ & & \swarrow & & \swarrow & & & \\ & & \mathbb{Z}/5\mathbb{Z} & & \mathbb{Z}/9\mathbb{Z} & \cdots & \cdots & \\ & & \vdots & & \vdots & & & \\ & & \mathbb{Z}/p\mathbb{Z} & & \vdots & & & \\ & & \vdots & & \vdots & & & \end{array} \right)$$

More concretely:

$$\hat{\mathbb{Z}} = \left\{ (a_n)_{n=1}^{\infty} \in \prod_{n=1}^{\infty} \mathbb{Z}/n\mathbb{Z} \mid j \mid i \Rightarrow a_i \equiv a_j \pmod{j} \right\}$$

$$\mathbb{Z} \hookrightarrow \hat{\mathbb{Z}}, \quad a \mapsto ([a]_n)_{n=1}^{\infty} \quad \pi_n: \hat{\mathbb{Z}} \rightarrow \mathbb{Z}/n\mathbb{Z} \quad (a_k)_{k=1}^{\infty} \mapsto a_n$$

$\underline{\text{Ex.}}$ $17 \mapsto (0, 1, 2, 1, 2, 5, 3, \dots$
 $\dots, 1, 0, 17, 17, 17, \dots)$

Consider $\hat{a} = (a_n)_{n=1}^{\infty}$, $a_n = \begin{cases} 1, & p|n \\ 0, & p \nmid n \end{cases}$
 (p fixed)

if $j|i$ is $a_i \equiv a_j \pmod{j}$?

Problem: Find $\hat{a} \in \hat{\mathbb{Z}} \setminus \mathbb{Z}$.

EX ② $\mathbb{K}[t] \cong \lim_{\leftarrow} \frac{\mathbb{K}[t]}{(t)^n} =$
 $= \lim_{\leftarrow} \left(\frac{\mathbb{K}[t]}{(t)} \leftarrow \frac{\mathbb{K}[t]}{(t)^2} \leftarrow \frac{\mathbb{K}[t]}{(t)^3} \leftarrow \dots \right)$
 $\pi_n : \mathbb{K}[t] \rightarrow \frac{\mathbb{K}[t]}{(t)^n}, \sum_{k=0}^{\infty} c_k t^k \mapsto \sum_{k=0}^{n-1} c_k t^k$

EX ③ R ring, $I \subset R$ ideal.

$$R_I := \lim_{\leftarrow} R/I^n = \lim_{\leftarrow} (R/I \leftarrow R/I^2 \leftarrow \dots)$$

R_I has a ring structure. $\left. \begin{array}{l} \text{in } \underline{Ab} \\ \text{OR: in } \underline{Rng} \end{array} \right\}$

Topology. X set

Def A topology on X is a subset $\mathcal{T} \subset \mathcal{P}(X)$ such that \mathcal{T} is a power set of X

1) $\emptyset, X \in \mathcal{T}$

2) if $U_\alpha \in \mathcal{T}, \alpha \in I \Rightarrow \bigcup_{\alpha \in I} U_\alpha \in \mathcal{T}$

3) if $U_1, \dots, U_n \in \mathcal{T} \Rightarrow U_1 \cap \dots \cap U_n \in \mathcal{T}$

Elements of \mathcal{T} are called open (wrt. \mathcal{T}).

A **topological space** is a set X together with a topology \mathcal{T} on X .

A map $f: X \rightarrow Y$ (X, Y top. spaces) is **continuous** if

$$\forall U \in \mathcal{T}_Y : f^{-1}(U) \in \mathcal{T}_X.$$

EX. The **discrete topology** on a set X is $\mathcal{T} = \mathcal{P}(X)$. Equivalently, it's the topology s.t. $\{x\} \in \mathcal{T} \quad \forall x \in X$.

Def If $\mathcal{A} \subset \mathcal{P}(X)$, the topology generated by \mathcal{A} , $\langle \mathcal{A} \rangle$, is the ^{unique} smallest topology containing \mathcal{A} :

$\langle \mathcal{A} \rangle = \{ \text{arbitrary unions of finite intersections of sets in } \mathcal{A} \}$

The limit topology on

$A = \lim A_j$ in $\mathcal{C} = \text{concrete category}$

is the topology on A with fewest open sets such that

$\pi_j : A \rightarrow A_j$ are all continuous, where i.e. $\exists \mathcal{C} \rightarrow \underline{\text{Set}}$

the A_j 's are given the discrete topology.
That is it's the topology generated
by $\mathcal{A} = \{ \pi_j^{-1}(\{x\}) \mid x \in A_j \}$

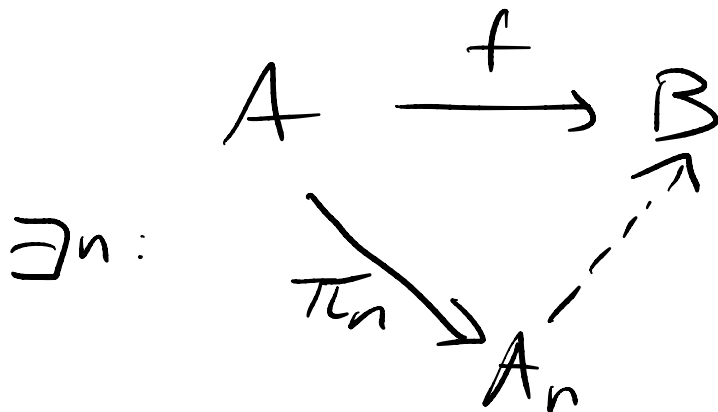
Say $\mathcal{C} = \frac{A_b}{j}$. (Then $\ker \pi_j$ are open
for all j) " $\pi_j^{-1}(\{0\})$ ".

Note A map $f: X \rightarrow Y$, Y discrete
is cont. iff f is locally constant
(Exercise). i.e. $\forall x \in X \exists U \in \mathcal{T}_x, x \in U: f|_U$ is constant.

$A = \lim A_j$ w/ limit top.

$\pi_j: A \rightarrow A_j$ continuous.

Then $f: A \rightarrow B$ is continuous
iff f factors through some π_j : ← discrete



Example A continuous group homomorphism

$$\widehat{\mathbb{Z}} \longrightarrow \mathbb{k}^\times (= GL(1, \mathbb{k}))$$

\iff a group hom. map $\mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{k}^\times$ for some n

{ Roots of unity }

\iff

1-dim'd continuous reps of $\widehat{\mathbb{Z}}$

$$\begin{array}{ccccccc} \varepsilon & \longrightarrow & \widehat{\mathbb{Z}} & \longrightarrow & \mathbb{Z}/n\mathbb{Z} & \longrightarrow & \mathbb{k}^\times \\ \varepsilon^n = 1 & & & & [1] & \longmapsto & \varepsilon \end{array}$$