Quantum Algebras

$\mathbb{k}$ field, $q \in \mathbb{k} \setminus \{0\}$ (Ex. $\mathbb{k} = \mathbb{Q}(q)$)

Example. (Quantum plane).

$$\mathbb{k}_q[x,y] = \mathcal{O}_q(\mathbb{A}^2_{\mathbb{k}}) = \mathbb{k}\langle x,y\rangle / \langle yx - qxy\rangle$$

We have $\mathbb{k}_q[x,y] \cong \mathbb{k}[x][y;\sigma,0]$

where $\sigma : \mathbb{k}[x] \to \mathbb{k}[x]$, $\sigma(x) = qx$

$$yx = \sigma(x)y \overset{\text{want}}{=\!=} qxy$$

$\Rightarrow$ GKdim $\mathbb{k}_q[x,y] = 1 + 1 = 2$

Claim

Basis for $\mathbb{K}_q[x,y]$ $\mathcal{B} = \{x^k y^l\}_{\substack{k \geq 0 \\ l \geq 0}}$

Pf Spans

$1, x, y \in \mathcal{B}$. Suffices to show

$\mathrm{Span}_{\mathbb{K}} \mathcal{B}$ is $\mathbb{K}_q[x,y]$-subbimodule

$$x \, x^k y^l = x^{k+1} y^l$$
$$y \, x^k y^l = q^k x^k y^l$$

& similarly on the right.

$\Rightarrow$ $\mathrm{Span}_{\mathbb{K}} \mathcal{B}$ is a subalgebra

$\Longrightarrow \operatorname{Span}_{\mathbb{k}} \mathcal{B} = \mathbb{k}_q[x,y]$

Linear independence. One reduction rule:

$$yx \longrightarrow qxy$$

No ambiguities to resolve

Diamond lemma $\Longrightarrow$ $\mathcal{B}$ lin. indep.

Center Let $A = \mathbb{k}_q[x,y]$.

$$Z(A) = \{ z \in A \mid za = az \ \forall a \in A \}$$

$$= \{ z \in A \mid zx = xz \ \& \ zy = yz \}$$

Write $z = \sum_{k,l \geq 0} c_{kl} x^k y^l$, $c_{kl} \in \mathbb{k}$

Then $zx - xz = 0 \iff \sum_{k,l \geq 0} c_{kl} (q^l - 1) \cdot x^{k+1} y^l$

$\iff (c_{kl} \neq 0 \implies q^l = 1)$

Similarly $zy = yz \iff (c_{kl} \neq 0 \implies q^k = 1)$

<u>So</u> $Z(A) = \begin{cases} \mathbb{k} & \text{if } q \text{ is not a root of unity} \\ \mathbb{k}[x^d, y^d] & \text{if } |\langle q \rangle| = d \end{cases}$

Remark If $|q| = d$ then A is free of finite rank over its center:

$$A = \bigoplus_{k,l \geq 0} \mathbb{k}\, x^k y^l =$$

$$= \bigoplus_{0 \leq k, l \leq d-1} Z(A) \cdot x^k y^l$$

Fact: This implies A is a PI-algebra:

polynomial identity.

$\exists f \in \mathbb{k}\langle T_1, T_2, \dots, T_N \rangle$, $f \neq 0$, s.t.

$$f(a_1, a_2, \dots, a_N) = 0 \quad \forall a_i \in A.$$

$\mathbb{K}_q[x_1, x_2, \dots, x_n]$?

General: $Q = (q_{ij})_{i,j=1}^n$ $n \times n$-matrix such that $q_{ii} = 1$, $q_{ij} q_{ji} = 1$

(Q is multiplicatively skew-symmetric)

(Special case: Take $q_{ij} = q^{s_{ij}}$ where $S = (s_{ij})_{i,j=1}^n$ is a skew symmetric integer matrix.)

$$A_Q = \mathbb{K}_Q[x_1, \dots, x_n] := \frac{\mathbb{K}\langle x_1, \dots, x_n \rangle}{\langle x_i x_j - q_{ij} x_j x_i \rangle}$$
$$\forall i, j$$

<u>Problem</u> Show A_Q is isomorphic to an iterated skew pol. ring:

$$\mathbb{K}[x_1][x_2; \sigma_2, 0] \cdots [x_n; \sigma_n; 0]$$

$$\mathrm{GKdim}\, A_Q = n$$

Quantum torus

Usual n-dim'l torus: $\overbrace{S^1 \times S^1 \times \cdots \times S^1}^{(S^1)^n}$

Complexified torus: $\mathbb{C}^\times \times \mathbb{C}^\times \times \cdots \times \mathbb{C}^\times$

$$\left[\begin{array}{ccccccc} (S^1)^n & \xrightarrow{\text{Lie alg}} & \mathbb{R}^n & \xrightarrow[\mathbb{C} \otimes_{\mathbb{R}} \mathfrak{g}]{\text{complexify}} & \mathbb{C}^n & \xrightarrow{\text{Exp}} & (\mathbb{C}^\times)^n \\ \text{Real Lie group} & & & & & & \text{complex Lie group} \end{array}\right]$$

Replace $\mathbb{C}$ by $\mathbb{k}$: $\boxed{(\mathbb{k}^\times)^n}$

Alg of regular fcns: $\mathcal{O}((\mathbb{k}^\times)^n) \cong \mathbb{k}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$

Def The quantum torus is

$$\mathbb{k}_Q[x_1^{\pm 1}, \dots, x_n^{\pm 1}] = \frac{\mathbb{k}\langle x_1, x_1^{-1}, \dots, x_n, x_n^{-1}\rangle}{\left\langle \begin{array}{l} x_i x_i^{-1} - 1,\ x_i^{-1}x_i - 1 \\ x_i x_j - q_{ij}\, x_j x_i \end{array} \quad \forall i,j \right\rangle}$$

Quantum groups

Assume $q^2 \neq 1$

(conn., simply conn.)
Lie groups $\longleftrightarrow$ Lie algebras $\longleftrightarrow$ enveloping algebras $U(\mathfrak{g})$

Define: $$U_q(sl_2) = \frac{\mathbb{k}\langle E, F, K, K^{-1}\rangle}{\langle \text{Rels} \rangle}$$

Rels:
$$\begin{cases} KK^{-1} = 1 = K^{-1}K \\ KE = q^2 EK \\ KF = q^{-2} FK \\ EF - FE = \dfrac{K - K^{-1}}{q - q^{-1}} \end{cases}$$

$q \to 1$? Set $q = e^{\hbar} \in \mathbb{F}[[\hbar]]$

(Assume $\mathbb{F}[[\hbar]] \subset \mathbb{k}$)

Set $K = e^{\hbar H} = \sum_{n=0}^{\infty} \frac{1}{n!} H^n \hbar^n$

$\in \mathbb{F}[H][[\hbar]]$

Substitute in relations:

$$\frac{K - K^{-1}}{q - q^{-1}} = \frac{e^{\hbar H} - e^{-\hbar H}}{e^{\hbar} - e^{-\hbar}} = \frac{\sinh(\hbar H)}{\sinh(\hbar)}$$

$$\lim_{\hbar \to 0} \frac{\sinh(\hbar H)}{\sinh \hbar} \overset{\text{l'Hopital}}{=} \lim_{\hbar \to 0} \frac{H \cosh(\hbar H)}{\cosh(\hbar)} = H$$

So $EF - FE = \dfrac{K - K^{-1}}{q - q^{-1}}$

$q \to 1 \iff \hbar \to 0$

$\longrightarrow$ $EF - FE = H$ (Rel from $U(sl_2)$)

$U_q(sl_2) \hookrightarrow U_\hbar(sl_2)$ topological algebra ($\hbar$-adic topology)

$$\begin{array}{c} U_\hbar(sl_2) \\ \downarrow \\ \dfrac{U_\hbar(sl_2)}{\langle \hbar \rangle} \cong U(sl_2) \end{array}$$

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A Guide to Quantum Groups