Gelfand-Kirillov dimension.

Example $\operatorname{GKdim}(R \rtimes \mathbb{Z}) = ?$

Recall: Fix a group homom. $\alpha : \mathbb{Z} \to \operatorname{Aut}(R)$
(α is determined by $\sigma := \alpha(1)$, then
$\alpha(n) = \sigma^n \ \forall n \in \mathbb{Z}$)

Skew grp alg wrt α

$$R \rtimes \mathbb{Z} = R \rtimes_{\alpha} \mathbb{Z} = \bigoplus_{n \in \mathbb{Z}} R \cdot t^n$$

with multiplication

$$(r t^n) \cdot (s t^m) = r \sigma^n(s) t^{n+m}$$

$\forall r, s \in R,\ n, m \in \mathbb{Z}$.

$R \rtimes \mathbb{Z}$ can be viewed as a quotient of an iterated skew polynomial ring:

$$R \rtimes \mathbb{Z} \cong R[t;\sigma,0][t';\hat{\sigma}^{-1},0]/(tt'-1)$$

$\hat{\sigma}^{-1}(t) = t$

pf: $\varphi: rt^n \longmapsto rt^n$, $n \in \mathbb{Z}, r \in R$

isomorphism of left R-modules

& $\varphi(rt^n st^m) = \varphi(rt^n)\varphi(st^m)$

$\Rightarrow$ alg isomorphism.

$$GK(R[t;\sigma,0][t';\sigma^{-1},0]) = 2 + GKdim(R)$$

$tt'-1$ is a left regular element of

$R[t \dots][t' \dots]$: Say

$$p(t,t') = \sum_{k,l} r_{k,l}\, t^k (t')^l \quad , r_{k,l} \in R.$$

$$p(t,t')(tt'-1) = 0.$$

But leading term (wrt lexicographical order on $\{t^k t^l\}$) in LHS is the same as that of $p(t,t') \Rightarrow \underline{\underline{p(t,t')=0}}$

By Thm last time,

$$\mathrm{GKdim}(R \rtimes \mathbb{Z}) \leq 2 + \mathrm{GKdim}(R) - 1$$
$$= 1 + \mathrm{GKdim}(R).$$

Consider the injective algebra map.

$$\begin{array}{ccc} R[t;\sigma,0] & \hookrightarrow & R \rtimes \mathbb{Z} \\ \cong & & \cong \\ \bigoplus_{n=0}^{\infty} R t^n & & \bigoplus_{n \in \mathbb{Z}} R t^n \\ r t^n & \longmapsto & r t^n \end{array}$$

$$\Rightarrow \mathrm{GKdim}(R[t;\sigma,0]) \leq \mathrm{GKdim}(R \rtimes \mathbb{Z})$$

$$\Rightarrow \quad 1 + \mathrm{GKdim}\, R \leq \mathrm{GKdim}\,(R \ast \mathbb{Z})$$

$$\Rightarrow \boxed{\mathrm{GKdim}\,(R \ast \mathbb{Z}) = 1 + \mathrm{GKdim}\, R}$$

<u>Localization</u> = inverting some elements

Prototype: $\mathbb{Z} \rightsquigarrow \mathbb{Q}$

Here we'll focus on <u>central</u> localizations

i.e. "denominators" $\in$ center of alg.

A alg$/\mathbb{k}$, $S \subseteq A \setminus \{0\}$ a subset

such that

1) $s \in Z(A)$ $(sa = as \ \forall a \in A)$ $\forall s \in S$

2) $\forall s \in S$: s is left ($\Leftrightarrow$ right) regular:

$$sa = 0 \Rightarrow a = 0.$$

3) $\forall s, t \in S$: $s \cdot t \in S$; $1_A \in S$

Define an equivalence relation on

$S \times A$ by $(s,a) \sim (t,b)$ iff $ta = sb$

$\{ s^{-1} a$ $\qquad s^{-1} a = t^{-1} b$

$\qquad ta = sb$

Define $S^{-1}A = (S \times A)/\sim$. Denote class of (s,a) by $s^{-1}a$ for $s\in S$, $a\in A$.

Check that

1) $S^{-1}A$ is an algebra wrt operations $(s^{-1}a)(t^{-1}b)=(st)^{-1}ab$

$(s^{-1}a)+(t^{-1}b)=(st)^{-1}(ta+sb)$

2) $A \hookrightarrow S^{-1}A$, $a \mapsto 1^{-1}a$

3) Image of any $s\in S$ is invertible in $S^{-1}A$.

4) If $\varphi: A \hookrightarrow B$ is alg map s.t.

any $\varphi(s)$ is invertible in B, then $\exists$ comm. diagram

$$\begin{tikzcd}
A \arrow[rr, hook] \arrow[dr, hook] & & B \\
& S^{-1}A \arrow[ur, dashed] &
\end{tikzcd}$$

<u>Theorem</u> In the above setting

$$\operatorname{GKdim}(S^{-1}A) = \operatorname{GKdim}(A)$$

<u>Pf.</u> see [Lenagan-Krause, Ch. 4]

Problem If $R(\sigma,t)$ is a GWA where t is a regular element of R. (Recall: $t \in Z(R)\setminus\{0\}$ always)

$$S := \langle \sigma^n(t) \mid n \in \mathbb{Z} \rangle_{\text{Monoid}}$$
$$= \left\{ \sigma^{n_1}(t)\sigma^{n_2}(t)\cdots\sigma^{n_k}(t) \;\middle|\; \begin{array}{l} n_i \in \mathbb{Z} \\ k \geq 0 \end{array} \right\}$$

Show that $S^{-1}(R(\sigma,t)) \cong (S^{-1}R) \rtimes \mathbb{Z}$

Application:

$$\mathrm{GKdim}(R(\sigma,t)) \overset{\text{By Thm}}{=} \mathrm{GKdim}\, S^{-1}(R(\sigma,t))$$

$$\overset{\text{By Problem}}{=} \mathrm{GKdim}\,(S^{-1}R) * \mathbb{Z}$$

$$\overset{\text{By Today}}{=} 1 + \mathrm{GKdim}(S^{-1}R)$$

$$\overset{\text{By Thm}}{=} 1 + \mathrm{GKdim}(R)$$

Aside $\varphi : A \to B$ surjective alg map.

& suppose $\mathrm{GKdim}(A) = \mathrm{GKdim}(B)$.

Let $I = \ker \varphi$. $B \cong A/I$ so I

can't contain any left or right regular elements.

<u>Thm</u> $A = \bigcup_{n=0}^{\infty} A_n$ filtered alg.

$$\left(\begin{array}{l} 1 \in A_0 \subseteq A_1 \subseteq \cdots\cdots, \quad A_n A_m \subseteq A_{n+m} \\ \mathrm{gr}\, A = \bigoplus_{n=0}^{\infty} A_n / A_{n-1} \qquad A_{-1} := 0 \end{array} \right)$$

Assume $\dim A_n < \infty \ \forall n \geq 0$ and that $\mathrm{gr}\, A$ is finitely generated. Then $\mathrm{GKdim}(A) = \mathrm{GKdim}(\mathrm{gr}\, A)$.

Application

$$\mathrm{GKdim}(U(\mathfrak{sl}_2)) = \mathrm{GKdim}(\mathbb{C}[\bar{e},\bar{f},\bar{h}]) = 3$$

More generally $\forall$ fd Lie alg $\mathfrak{g}$:

$$\mathrm{GKdim}\, U(\mathfrak{g}) = \mathrm{GKdim}\, \mathrm{gr}\, U(\mathfrak{g}) \overset{\mathrm{PBW}}{=} \mathrm{GKdim}\, S(\mathfrak{g}) = \dim \mathfrak{g}.$$

symm. alg on $\mathfrak{g}$