Gelfand-Kirillov Dimension.

alg = f.g. alg.

If A f.g. $\mathbb{K}$-alg we put

$$\mathrm{GKdim}(A) = \overline{\lim}\left(\log_n d_V(n)\right),$$ where

$V \subset A$ is a f.d. generating subspace.

Lemma a) $\mathrm{GKdim}(A) = \inf\{\rho \in \mathbb{R}_{\geq 0} : d_V(n) \leq n^\rho, n \gg 0\}$

$$= \inf\{\rho \in \mathbb{R}_{\geq 0} : \mathcal{G}(A) \leq \mathcal{P}_\rho\}$$

b) If $\mathcal{G}(A) = \mathcal{G}(B)$ then $\mathrm{GKdim}(A) = \mathrm{GKdim}(B)$

Existence (Warfield) $\forall r \geq 2$ $\exists$ a two-generated algebra $A = \mathbb{K}\langle x, y\rangle / I$ such that $\mathrm{GKdim}(A) = r$.

Constructions.

Proposition 1. If B is a subalgebra or homomorphic image of A, then
$$\mathrm{GKdim}(B) \leq \mathrm{GKdim}(A)$$

Proof. V = f.d. generating subspace of A.

W = ——— " ——— of B

Say B is a subalg of A.

Then $V+W$ is f.d. and $W \subset V+W$

$$d_W(n) \leq d_{V+W}(n)$$

$$\left(\begin{array}{l}\text{Recall: } d_V(n) = \dim(V^0+V^1+\dots+V^n) \\ \& \ V^0+V^1+\dots+V^n = V^n \text{ if } 1_A \in V\end{array}\right)$$

$$\Rightarrow \overline{\lim} \log_n d_W(n) \leq \overline{\lim} \log_n d_{V+W}(n)$$

$$\mathrm{GKdim}(B) \leq \mathrm{GKdim}(A).$$

If B is a homomorphic image of A say $\varphi : A \twoheadrightarrow B$ onto alg map

Let $W = \varphi(V)$, $V \subset A$ f.d gen. subsp.

$W^n = \varphi(V^n) \Rightarrow \dim W^n \leq \dim V^n$

..... Q.E.D.

<u>Proposition</u> 2.

$$\mathrm{GKdim}\,(\overset{A_1 \oplus A_2 \text{ as v.sp.}}{\downarrow}{A_1 \times A_2}) = \max\{\mathrm{GKdim}\, A_i \mid i=1,2\}$$

<u>Proof</u> $\pi_i : A_1 \times A_2 \to A_i$, $i = 1,2$

π_i are epimorphisms so $(\geq)$ holds.

$(\leq)$: Let $\gamma = \mathrm{RHS}$. If $\gamma = \infty$, $\leq$ holds.

Let $W \subset A_1 \times A_2$ be a f.d. generating subspace, $U_i = \pi_i(W)$. Then $W \subseteq U_1 \oplus U_2$, $W^n \subseteq U_1^n \oplus U_2^n$

Let $\varepsilon > 0$. Then

$$d_{U_i}(n) \leq n^{\gamma + \varepsilon/2}, \quad i = 1, 2 \quad \forall n \gg 0$$

(by Lemma). We have

$$d_W(n) \leq d_{U_1}(n) + d_{U_2}(n) \leq 2n^{\gamma + \varepsilon/2} < n^{\varepsilon/2} \cdot n^{\gamma + \varepsilon/2} = n^{\gamma + \varepsilon}, \quad n \gg 0.$$

By Lemma we get $\leq$. QED.

Corollary 3. $\operatorname{GKdim} A/_{I_1\cap\cdots\cap I_n} = \max_i \operatorname{GKdim} \frac{A}{I_i}$ where $I_i \subseteq A$ are ideals.

Proof

$$A/_{I_1\cap\cdots\cap I_n} \hookrightarrow \frac{A}{I_1} \times \cdots \times \frac{A}{I_n}$$

shows that ($\leq$) holds by Prop 1, 2.

On the other hand

$$A/_{I_1\cap\cdots\cap I_n} \twoheadrightarrow A/_{I_i} \quad \forall i$$

shows that ($\geq$) holds by Prop 1.

QED.

Skew Polynomial Rings (Ore extensions.)

$$B = A[x; \sigma, \delta]$$

- A $\Bbbk$-alg
- $\sigma \in \mathrm{Aut}_{\Bbbk}(A)$,
- $\delta : A \longrightarrow A$ $\Bbbk$-linear map

$$\delta(ab) = \delta(a)b + \sigma(a)\delta(b)$$

$\forall a, b$ ($=\delta$ is a σ-derivation).

$$B = \bigoplus_{n=0}^{\infty} A \cdot x^n \text{ as left } A\text{-modules}$$

such that $x \cdot a = \sigma(a)x + \delta(a)$ $\forall a \in A$.

a=t

Ex. $A_1(\mathbb{k}) = \mathbb{k}\langle t, \partial\rangle / (\partial t - t\partial - 1)$

$$\cong \underbrace{\mathbb{k}[t]}_{A}[\partial; \mathrm{Id}_A, \delta = \tfrac{d}{dt}]$$

$$\partial \cdot t = t \cdot \partial + 1$$

Prop $\operatorname{GKdim}(B) = 1 + \operatorname{GKdim}(A)$.

proof $V \subset A$ f.d. gen. subsp., $1_A \in V$.

Put $W = V + \mathbb{k}x$. Then

$$V^n + V^n x + \cdots + V^n x^n \subset (V + \Bbbk x)^{2n} = W^{2n}$$

so $(n+1)\, d_V(n) \leq d_W(2n)$

$$\begin{aligned} \operatorname{GKdim}(B) &\geq \overline{\lim} \log_n d_W(2n) \\ &\geq \overline{\lim} \log_n (n+1)\, d_V(n) \\ &= \lim_{n \to \infty} \log_n (n+1) + \\ &\quad + \overline{\lim} \log_n d_V(n) \\ &= 1 + \operatorname{GKdim} A. \end{aligned}$$

($\leq$) proof skipped. Q.E.D.

Cor $\operatorname{GKdim}(A[x_1,\dots,x_n]) = n + \operatorname{GKdim}(A)$.

Quotients

Prop $I \subseteq A$ ideal, such that I contains an element c which is left or right regular

$a \cdot c = 0 \Rightarrow a = 0$ $\quad c \cdot a = 0 \Rightarrow a = 0$

then $\operatorname{GKdim}(A/I) \leq \operatorname{GKdim}(A) - 1$

GWA $R(\sigma, t) = \langle X, Y \rangle_{R\text{-ring}} \Big/ \begin{pmatrix} YX - t \\ XY - \sigma(t) \\ Xr - \sigma(r)X \\ Yr - \sigma^{-1}(r)Y \\ \forall r \in R \end{pmatrix}$

$\begin{cases} R \text{ integral domain} & t \in R \setminus \{0\} \\ & \sigma \in \mathrm{Aut}_{\Bbbk}(R) \end{cases}$

$\hat{\sigma}(Y) = Y$

$\hat{\sigma}|_R = \sigma$

We saw $R(\sigma, t) \cong \dfrac{R[Y; \sigma^{-1}, 0][X; \hat{\sigma}, \delta]}{(YX - t, XY - \sigma(t))}$

$\cancel{XY = YX + \delta(Y)}$

$\begin{cases} \delta(Y) = \sigma(t) - t \\ \delta(r) = 0 \end{cases}$

$\boxed{\mathrm{GKdim}\, R(\sigma, t) \ ?}$

Problem