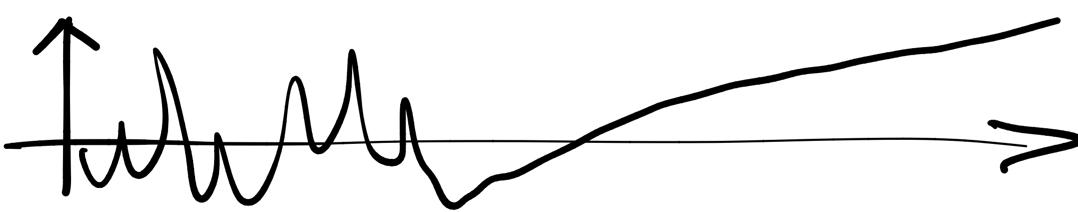
# GK Dimension II.

**Growth** Motivation: $A$ $\mathbb{k}$-alg, $V \subseteq A$ f.d. generating subspace we put

$$d_V(n) = \dim_{\mathbb{k}} A_n \quad , \quad A_n = \underbrace{V^0}_{=\mathbb{k}} + V^1 + V^2 + \dots + V^n$$

$d_V : \mathbb{N} \to \mathbb{R}$. Want to compare such fcns.

$$\Phi = \{ f : \mathbb{N} \to \mathbb{R} \mid f \text{ is eventually monotone increasing positive-valued} \}$$



$n \leq m \Rightarrow f(n) \leq f(m)$

**Def** 1) For $f, g \in \Phi$ define $f \leq^* g$ if $\exists\, c, m \in \mathbb{N}$ : $f(n) \leq c \cdot g(m \cdot n)$, $n \gg 0$.

2) Say $f \sim g$ if $f \leq^* g$ & $g \leq^* f$.

3) $\mathcal{G}(f) = [f]$ equivalence class of $f$ is called the growth of $f$.

4) $\leq^*$ induces a partial order on $\Phi/\sim$ denoted $\leq$

## Examples

$\cancel{c_a > 0}$
$\cancel{d_b > 0}$

(1) $f$, $g$ polynomials

$$f(n) = 1 n^a + c_{a-1} n^{a-1} + \cdots + c_0 \ , \quad c_i \in \mathbb{R}$$

$$g(n) = 1 n^b + \cdots + d_0 \ , \quad d_i \in \mathbb{R}$$

$a = \deg f \quad b = \deg g \qquad f, g \in \Phi.$

Then $f \sim g$ iff $a = b$. Suppose

$$f(n) \leq^* C\, g(mn) \quad n \gg 0$$

$$1 n^a + \cdots \leq C \cdot (1 m^b n^b + \cdots) \ , \quad n \gg 0$$

$$\lim_{n\to\infty} \frac{n^{a} + \cdots}{m^{b} n^{b} + \cdots} \leq C \implies \underline{a \leq b}$$

$\underline{\text{So}}$ $f \sim g \implies a = b$. (Converse easy).

For $\gamma \in \mathbb{R}_{\geq 0}$ we define $\mathcal{P}_\gamma$ to be the growth of $n \longmapsto n^{\gamma}$.

(2) $\forall \varepsilon \in \mathbb{R}_{>0}$ let $\mathcal{E}_\varepsilon$ be the growth of the function $n \longmapsto e^{n^{\varepsilon}}$

$\underline{\text{Fact}}$: $\varepsilon < \eta \iff \mathcal{E}_\varepsilon < \mathcal{E}_\eta$

(3) $f(n) = \log n$. Then $\mathcal{G}(f) > \mathcal{P}_0$ but $\mathcal{G}(f) < \mathcal{P}_\varepsilon$ $\forall \varepsilon > 0$.

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**Lemma** If $V, W$ are two fin-dim'l generating subspaces of an alg $A/\mathbb{k}$ then $\mathcal{G}(d_V) = \mathcal{G}(d_W)$.

Proof $W \subset V^0 + V^1 + \cdots + V^s$ for some $s > 0$

$V \subset W^0 + W^1 + \cdots + W^t$ — " — $t > 0$

We have $W^0 + W^1 + W^2 + \cdots + W^n \subseteq$

$$\subseteq \sum_{k=0}^{n} \left( \sum_{\ell=0}^{s} V^{\ell} \right)^{k} \subset V^0 + V^1 + \dots + V^{sn}$$

$$\Rightarrow d_W(n) \le d_V(sn) \quad \forall n \ge 0$$

$\Rightarrow d_W \le^* d_V$ By symmetry $d_V \le^* d_W$

$\Rightarrow d_V \sim d_W \Rightarrow \mathcal{G}(d_V) = \mathcal{G}(d_W)$ Q.E.D.

<u>Def</u> the growth of A is defined as $\mathcal{G}(A) := \mathcal{G}(d_V)$ where $V$ is any fin. dim'l generating subspace.

Ex. A f.g. alg has

polynomial growth if $\mathcal{G}(A) = \mathcal{P}_m$, $m \in \mathbb{N}$

exponential growth if $\mathcal{G}(A) = \mathcal{E}_1$

subexponential growth if $\mathcal{G}(A) < \mathcal{E}_1$ yet $\forall m \in \mathbb{N} : \mathcal{G}(A) \nleq \mathcal{P}_m$

Ex. $\mathcal{G}(\mathbb{k}\langle x, y\rangle) = \mathcal{E}_1$

Ex. (M. Smith) $A = \mathbb{k}\langle x, y_1, y_2, y_3, \dots\rangle / I$ where $I = \langle x y_i - y_i x - y_{i+1},\ y_i y_j - y_j y_i\ \forall i,j \rangle$

$V = \mathbb{k}x + \mathbb{k}y_1$ Then $d_V(n) \sim \exp(\sqrt{n})$.

So $A$ has subexponential growth.

This shows $\exists$ algebras $A$ with $\text{GK-dim}\, A = \infty$ yet $A$ is not a free algebra.

(Recall $\text{GKdim}\, A = \limsup_n \left(\log_n d_V(n)\right)$)

class of bounded fcn.

**Note** $\mathcal{G}(A) = \mathcal{P}_0$ iff $\dim_{\mathbb{k}} A < \infty$.

Pf: ($\Longrightarrow$): If $\dim(V^0 + V^1 + \dots + V^n)$ is bounded then $\exists n > 0$ : $V^{n+1} \subset V^0 + \dots + V^n$

($\Longleftarrow$) Take $V = A$.

QED.

**Proposition** [Borho-Kraft] If $A$ is fin. gen. but not finite-dimensional, then

$$\mathcal{P}_1 \leq \mathcal{G}(A) \leq \mathcal{E}_1$$

**Proof** Assume WLOG $1_A \in V$. Then

$$V^0 + V^1 + \cdots + V^n = V^n$$

$$\left( \text{Ex. } a_1 a_2 = a_1 a_2 \cdot \underbrace{1 \cdot 1 \cdots 1}_{n-2} \in V^n \quad V = \{1, a_1, a_2, \ldots\} \quad V^2 \right)$$

$$\Rightarrow d_V(n) = \dim(V^n) \leq \dim(V^{\otimes n}) = (\dim V)^n$$

exponential fcn of $n$

$$V^n \leftarrow V^{\otimes n}$$

$$a_1 a_2 \cdots a_n \longleftarrow\!\shortmid\ a_1 \otimes \cdots \otimes a_n$$

$$\Rightarrow \mathcal{G}(A) = \mathcal{G}(d_V) \leq \mathcal{E}_1$$

Also $V^n \subsetneq V^{n+1} \subsetneq V^{n+2} \subsetneq \cdots$

since $\dim A = \infty$

$$\Rightarrow \dim V^n \geq n = n^1$$

$$\Rightarrow \mathcal{G}(A) = \mathcal{G}(d_V) \geq \mathcal{P}_1$$

QED

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Problem Find the growth of

$$\mathbb{k}[x]\langle y, z\rangle = \boxed{\mathbb{k}\langle X, y, z\rangle \Big/ \langle Xy - yX,\ Xz - zX\rangle}$$

$$\left( R\langle X, Y\rangle \quad \langle X, Y\rangle_{R\text{-alg}} \right.$$

$$\text{or} \quad \langle X, Y\rangle_{R\text{-ring}}$$

$$\left. \text{GWAs} \right)$$