Discussion

G-Crossed product

1) $A = \bigoplus_{g \in G} A_g$ graded alg

2) $\forall g \in G$, A_g contains an element which is invertible in A.

2) $\Longleftrightarrow$ 2') $\forall g \; \exists a_g \in A_g \cap A^\times$ such that

$a_g \cdot r = \sigma_g(r) a_g$ $\forall r \in A_e$ where $\sigma : G \to Aut(A_e)$ is a map

& $a_g a_h = \alpha(g,h) a_{gh}$ where $\alpha : G \times G \to A_e \cap A^\times$

2) $\Leftarrow$ 2') Obvious.

2) $\Rightarrow$ 2') Pick any $a_g \in A_g \cap A^\times$.

$$a_g a_h r = a_g \sigma_h(r) a_h = \sigma_g(\sigma_h(r)) \underline{a_g a_h}$$

$$\alpha(g,h) a_{gh} r = \alpha(g,h) \sigma_{gh}(r) a_{gh} =$$

$$= \alpha(g,h) \sigma_{gh}(r) \alpha(g,h)^{-1} \underbrace{\alpha(g,h) a_{gh}}_{= a_g a_h}$$

$$\boxed{\sigma_g \sigma_h(r) = \underline{\alpha(g,h)}\, \sigma_{gh} \underline{(r)\, \alpha(g,h)^{-1}}}$$

Growth of algebras.

A algebra / $\mathbb{k}$ - field.

$V \subset A$ finite-dim'l generating subspace.

Put $V^0 = \mathbb{k}1_A$, $V^1 = V$, $V^2 = V \cdot V$, ...

$= \operatorname{Span}\{xy \mid x, y \in V\}$

$V^n = \operatorname{Span}\{x_1 x_2 \cdots x_n \mid x_i \in V\}$

Equivalently, V is the $\mathbb{k}$-span of a finite generating set for A.

The growth of A (with respect to V) is defined to be

$$f(n) = \log_n\left(\dim \sum_{k=0}^{n} V^k\right) = \frac{\log\left(\dim \sum_{k=0}^{n} V^k\right)}{\log n}$$

Ex. $A = \Bbbk[x]$, $V = \Bbbk x$

$$\cancel{\dim V^n = \dim \Bbbk x^n = 1}$$

$$\dim \sum_{k=0}^{n} V^k = \dim\left(\Bbbk 1 + \Bbbk x^1 + \dots + \Bbbk x^n\right)$$

$$= n+1$$

polynomial in n of degree 1

$$f(n) = \frac{\log(n+1)}{\log n}$$

Ex. $A = \mathbb{k}\langle x, y\rangle$ free alg on 2 gens.

$$V = \mathbb{k}x + \mathbb{k}y$$

$$\dim\left(\sum_{k=0}^{n} V^k\right) = \#\text{words in } \{x,y\} \text{ of length} \le n =$$

$$= 1 + 2 + 2^2 + \cdots + 2^n$$

$$= 2^{n+1} - 1$$

exponential function in n

$$f(n) = \frac{\log(2^{n+1}-1)}{\log n}$$

Ex. $A = \mathbb{k}[x_1, \dots, x_k]$

$$V = \mathbb{k}x_1 + \mathbb{k}x_2 + \cdots + \mathbb{k}x_k$$

$$\dim V^n = \binom{n+k-1}{k-1} = \text{pol. in } n \text{ of deg } k-1$$

$$x_1 \mid x_2 \mid \cdots \mid x_k$$

$$\dim V^0 + V^1 + \cdots + V^n$$

$$= \text{pol. in } n \text{ of deg } k$$

$$= \sum_{j=0}^{n} \binom{j+k-1}{k-1} = * n^k + \cdots$$

Next, take limit as $n \to \infty$.
May not exist, take limsup instead

<u>Def</u> The Gelfand-Kirillov dimension of a finitely generated algebra A is defined as

$$\operatorname{GKdim}(A) = \limsup \left(\log_n \dim \sum_{k=0}^{n} V^k \right)$$

where V is a choice of finite-dim'l generating subspace.

Lemma 1 Independent of choice of V.

$$W \subset V^{(n)}, \quad V \subset W^{(m)}$$

$$V^{(n)} = \sum_{k=0}^{n} V^k \qquad A = \bigcup_{n=0}^{\infty} V^{(n)} = \bigcup_{n=0}^{\infty} W^{(n)}$$

Def For a not necessarily finitely generated algebra A we define

$$\operatorname{GKdim} A = \sup_{V \subset A,\ \dim V < \infty} \left(\limsup \left(\log_n \dim V^{(n)} \right) \right)$$

<u>Ex</u>. $\operatorname{GKdim} \Bbbk\langle x, y \rangle \approx \lim_{n \to \infty} \left(\log_n 2^n \right) =$

$$= \lim_{n \to \infty} \underbrace{\frac{n}{\log n}}_{\to \infty} \cdot \log 2 = \infty$$

<u>Teaser</u>: 1) If A is commutative then $\operatorname{GKdim} A =$ <u>Krull-dim</u>(A)

$\sup \{ n \mid \exists\ P_0 \subset P_1 \subset \dots \subset P_n \text{ in } \operatorname{Spec}(A) \}$

2) $\operatorname{Gdim} A \in [0,1] \cup [2,\infty)$

gap proved by Bergman.

3) Constructions

- $A[x;\sigma,\delta]$
- A/I

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