Examples of Clifford theory ideas

R integral domain$/\mathbb{k}$ (ex. $\mathbb{C}[x]$

$\sigma \in \mathrm{Aut}_{\mathbb{k}}(R)$ $\sigma(p(x)) = p(x-1)$)

General fact: 1-1 correspondence between

$$\left\{\begin{array}{c}\text{simple } A\text{-modules} \\ + \text{ nonzero elt.}\end{array}\right\} \longleftrightarrow \{\text{maximal left ideals} \subsetneq A\}$$

$A = \mathbb{k}$-alg

$$(M, v) \longmapsto \mathrm{ann}_A(v) = \{a \in A \mid a.v = 0\}$$

$$(A/I, 1+I) \longleftarrow I$$

$\mathrm{Ann}_A(M) = \{a \in A \mid aM = 0\}$ 2-sided ideal

R commutative

$$\Rightarrow \left\{\begin{array}{c}\text{simple } R\text{-modules} \\ + \text{ nonzero elt.}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{maximal ideals} \\ \text{of } R\end{array}\right\}$$

$$(R/\mathfrak{m}, 1+\mathfrak{m}) \longleftarrow\!\shortmid \ \mathfrak{m}$$

Ex.

Nullstellensatz $\Rightarrow$ if $\overline{\mathbb{k}} = \mathbb{k}$ then any max'l ideal of $\mathbb{k}[x_1, \ldots, x_n]$ has the form

$$\mathfrak{m} = \mathfrak{m}_\lambda = (x_1 - \lambda_1, x_2 - \lambda_2, \ldots, x_n - \lambda_n)$$

$$\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{k}^n$$

Note $\mathfrak{m}_\lambda . v = 0$ $\quad v \in M$ R-module

$$\Updownarrow \qquad R = \Bbbk[x_1, \dots, x_n]$$

$$(x_i - \lambda_i).v = 0 \quad \forall i$$

$$\Longleftrightarrow \quad x_i . v = \lambda_i v \quad \forall i$$

Def R comm ring, M R-module.
A nonzero vector $v \in M$ is a weight vector if $\mathfrak{m} v = 0$ for some maximal ideal $\mathfrak{m} \subset R$.

R integral domain$/\mathbb{k}$

$\sigma \in \mathrm{Aut}_{\mathbb{k}}(R)$ infinite order automorphism.

$$A = R \rtimes \langle \sigma \rangle_{grp} = R \# \mathbb{k}\langle \sigma \rangle_{grp}$$
$$= R \# \mathbb{k}[\sigma, \sigma^{-1}]$$

$$\Rightarrow A = \bigoplus_{n \in \mathbb{Z}} R\sigma^n = \bigoplus_{n \in \mathbb{Z}} \sigma^n \cdot R$$

$$\sigma \cdot r = \sigma(r) \cdot \sigma$$

Have $R \longrightarrow A$, $r \longmapsto r 1_A$.

A is a $\mathbb{Z}$-crossed product:

$$A = \bigoplus_{n \in \mathbb{Z}} A_n \quad , \quad A_n = R\sigma^n$$

$$A_n A_m \subset A_{n+m}$$

$$\forall n \ \exists\, a_n \in A_n \cap A^\times : \ a_n = \sigma^n$$

$\mathbb{Z}$ is an infinite group!

Not true that any simple A-module is semisimple as R-module!

Look at subcategory instead!

Def The category of all (R-)weight A-modules is the cat. of all A-modules M that are semisimple as R-modules:

$$M = \bigoplus_{\mathfrak{m} \in \operatorname{MaxSpec}(R)} M_{\mathfrak{m}}$$

— set of maximal ideals of R

where $M_{\mathfrak{m}} = \{ v \in M \mid \mathfrak{m} v = 0 \}$.

Note. If M is semisimple as R-module, then $M = \bigoplus_i M_i$

simple R-modules, hence $M_i \cong R/\mathfrak{m}_i$ some $\mathfrak{m}_i$

Ex. $R = \mathbb{C}[x]$.

M = R-weight A-module

$$\Rightarrow \quad M = \bigoplus_{\mathfrak{m} \in \mathrm{MaxSpec}(R)} M_{\mathfrak{m}}$$

$$\left\{ \mathrm{MaxSpec}(R) = \{ \mathfrak{m}_\lambda = (x-\lambda) \mid \lambda \in \mathbb{C} \} \right\}$$

$$M = \bigoplus_{\lambda \in \mathbb{C}} M_{\mathfrak{m}_\lambda}$$

$$xv = \lambda v \; !$$

$$M_{\mathfrak{m}_\lambda} = \{ v \in M \mid \mathfrak{m}_\lambda v = 0 \} = E_\lambda(x 1_M)$$

$\mathbb{C}[x]$-weight A-module $=$

$=$ A-module on which x is diagonalizable.

Ex. $B \in M_n(\mathbb{C})$ gives rise to an n-dimensional module over $\mathbb{C}[x]$:

$$x.v \overset{\text{def}}{=} Bv \quad \forall v \in \mathbb{C}^n$$

$$\Longrightarrow p(x).v = p(B)v \quad \forall v \in \mathbb{C}^n$$

For ex $Bw = \lambda w \Leftrightarrow (x-\lambda).w = 0$

$$\pi' : \{x\} \longrightarrow M_n(\mathbb{C}) = \mathrm{End}(\mathbb{C}^n)$$
$$x \longmapsto B$$

$$\Rightarrow \pi : \mathbb{C}[x] \longrightarrow M_n(\mathbb{C}) \quad \text{alg hom.}$$

$$p(x) . v = \pi(p(x)) v$$
$$= p(\pi(x)) v = p(B) v$$

$\sigma \in \mathrm{Aut}_k(R)$. Then σ acts on $\mathrm{MaxSpec}(R)$ by:

$\sigma . \mathfrak{m} = \{ \sigma(a) \mid a \in \mathfrak{m} \}$.

Exercise: $\sigma . \mathfrak{m} \in \mathrm{MaxSpec}(R)$.

Thm

$A = R \# \Bbbk\langle\sigma\rangle_{grp}$ skew grp alg.

Let $\mathfrak{m} \in \mathrm{MaxSpec}(R)$ such that

$$\begin{aligned} \mathbb{Z} &\hookrightarrow \mathrm{MaxSpec}(R) \\ n &\longmapsto \sigma^n.\mathfrak{m} \end{aligned}$$

(Then $\langle\sigma\rangle.\mathfrak{m}$ is a torsionfree orbit)

There is a unique simple R-weight A-module M with $M_{\mathfrak{m}} \neq 0$.

Pf Take $M = \mathrm{Ind}_R^A(R/\mathfrak{m}) = A \otimes_R (R/\mathfrak{m})$.

Then $M = \bigoplus_{n\in\mathbb{Z}} (\sigma^n R) \otimes_R (R/\mathfrak{m})$

$$\sigma^n R \otimes_R R/m \cong (\mathbb{k}\sigma^n \otimes_{\mathbb{k}} R) \otimes_R (R/m)$$

$$\cong \mathbb{k}\sigma^n \otimes_{\mathbb{k}} \cancel{(R \otimes_R} R/m)$$

$$\cong \mathbb{k}\sigma^n \otimes_{\mathbb{k}} (R/m)$$

We have $\sigma^n(m) \cdot \sigma^n \otimes (1+m) =$

$$= \sigma^n(m)\,\sigma^n \otimes (1+m) = \sigma^n \cdot m \otimes (1+m)$$

$$= \sigma^n \otimes (0+m)$$

$$= 0$$

$$\sigma \cdot r = \sigma(r) \cdot \sigma$$

$$\Rightarrow \sigma^n \cdot r = \sigma^n(r) \cdot \sigma^n$$

$$\Longrightarrow M = \bigoplus_{n\in\mathbb{Z}} M_{\sigma^n(m)} \quad \text{weight module.}$$

$$M_{\sigma^n(m)} = R\sigma^n \otimes_R R/m$$

<u>Problem</u> Show M is simple as an A-module. Assume $R = \mathbb{C}[x]$ if necessary.

Problem Let V be a complex vector space and $L: V \to V$ a linear map. Make V into a $\mathbb{C}[x]$-module by defining $p(x).v = p(L)v$ $\forall v \in V$. Prove that V is a semisimple $\mathbb{C}[x]$-module iff L is diagonalizable.