CLIFFORD THEORY

(M. Lorenz, "A Tour of RepTh." § 3.6.4)

Recall: Γ group. A Γ-crossed product is a Γ-graded algebra A:

$$A = \bigoplus_{x \in \Gamma} A_x \quad \text{(as v. spaces)}$$

& $A_x A_y \subseteq A_{xy}$, $1_A \in A_{1_\Gamma}$

such that $\forall x \in \Gamma$: A_x contains an invertible element (unit) of A.

Note: If $u_x \in A_x$ are units $\forall x \in \Gamma$, $\forall a \in A_x$: $a = \underbrace{a (u_x)^{-1}}_{\in A_x A_{x^{-1}} \subset A_1} u_x \in A_1 \cdot u_x$

Let $B = \bigoplus_{x \in \Gamma} B_x$ be a Γ-crossed product, put $A = B_1$

Q: Relationship between A-modules and B-modules?

Ex. G group, $N \trianglelefteq G$ normal subgrp

Let $\Gamma = G/N$. Then:

$$\Bbbk G = \bigoplus_{g \in G} \Bbbk g = \bigoplus_{x \in \Gamma} \Bbbk \bar{x} N$$

where $\{\bar{x} \mid x \in \Gamma\} \subset G$ is a set of representatives for the cosets.

$B = \Bbbk G$, $A = \Bbbk N$ (identity component)

Twisting. If $B = \bigoplus_{x \in \Gamma} B_x$ is a Γ-crossed product, and $A = B_1$ and W an A-module, then we define ${}^x W = B_x \otimes_A W$ $\forall x \in \Gamma$.

and define

$$\Gamma_W = \{x \in \Gamma \mid {}^x W \cong W\}$$ "stabilizer of W"

Lemma: $\Gamma_W \leq \Gamma$, and ${}^x W \cong {}^y W$ iff $x \Gamma_W = y \Gamma_W$.

<u>Definition.</u> Let V be an A-module of finite length: $\exists$ seq of A-submods

$$0 = V_0 \subset V_1 \subset \cdots \subset V_\ell = V$$

such that V_i/V_{i-1} is simple $\forall i$

then for a simple A-module S,

$\ell_S := \#\{ i \mid V_i/V_{i-1} \cong S \}$ is the multiplicity of S in V.

Furthermore, if V is semisimple:

$$V \cong \bigoplus_{[S]} S^{\oplus \ell_S}$$

— sum over all isoclasses of simple A-modules.

then $V(S) := S^{\oplus \ell_S} \hookrightarrow V$

is the S-homogeneous component of S in V.

<u>Thm</u> (Clifford's Thm) Suppose $|\Gamma|<\infty$.
B Γ-crossed product, $A=B_1$.

For any simple B-module V, the restriction $\mathrm{Res}_A^B V$ is semisimple and of finite length. More precisely for any simple A-submodule S of $\mathrm{Res}_A^B V$ we have:

$$V = \bigoplus_{x\in\Gamma/\Gamma_S} ({}^{x}S)^{\oplus \ell_S}$$

Lastly, putting $B_S = \bigoplus_{x\in\Gamma_S} B_x \subseteq B$, $V(S)\subseteq \mathrm{Res}_{B_S}^B V$ and $V\cong \underbrace{\mathrm{Ind}_{B_S}^B V(S)}_{B\otimes_{B_S} V(S)}$.

Ex $\Gamma = S_2$, put $\begin{cases} 1 = (1) \\ \sigma = (12) \end{cases}$

$$B = \mathbb{C}[x] \rtimes S_2 \qquad \sigma . x = -x$$

$$\begin{cases} \sigma x = -x\sigma \text{ in } B \\ \sigma^2 = 1 \end{cases} \qquad \Rightarrow \sigma(p(x)) = p(-x)$$

$$B = \underbrace{\mathbb{C}[x] 1}_{= A \,\cong\, \mathbb{C}[x]} \oplus \mathbb{C}[x]\sigma$$

Isoclasses of simple A-modules are in bijection with $\mathbb{C}$:

$$\mathbb{C} \ni \lambda \rightsquigarrow \mathbb{C}_\lambda = \mathbb{C} \cdot 1_\lambda \qquad x \cdot 1_\lambda = \lambda 1_\lambda$$

$$(\Rightarrow p(x) 1_\lambda = p(\lambda) 1_\lambda)$$

${}^1\mathbb{C}_\lambda \cong \mathbb{C}_\lambda$ while ${}^\sigma\mathbb{C}_\lambda = \mathbb{C}[x]\sigma \otimes_{\mathbb{C}[x]} \mathbb{C}_\lambda =$

$$= \sigma \mathbb{C}[x] \otimes_{\mathbb{C}[x]} \mathbb{C}_\lambda$$

$$\sigma \mathbb{C}[x] \overset{v.sp.}{\cong} \mathbb{C}\sigma \otimes_{\mathbb{C}} \mathbb{C}[x]$$

$$= \mathbb{C}\sigma \otimes_{\mathbb{C}} \mathbb{C}_\lambda$$

$$= \mathbb{C} \cdot (\sigma \otimes 1_\lambda)$$

and $x \cdot (\sigma \otimes 1_\lambda) = x\sigma \otimes 1_\lambda =$

$$= -\sigma x \otimes 1_\lambda = -\sigma \otimes x 1_\lambda =$$

$$= (-\lambda) \cdot \sigma \otimes 1_\lambda$$

$\Longrightarrow \quad {}^{\sigma}\mathbb{C}_\lambda \cong \mathbb{C}_{-\lambda}$

Put $\Gamma_\lambda = \Gamma_{\mathbb{C}_\lambda}$. Then

$$\Gamma_\lambda = \begin{cases} 1, & \lambda \neq 0 \\ \Gamma = S_2, & \lambda = 0 \end{cases}$$

and $B_\lambda := B_{\mathbb{C}_\lambda} = \begin{cases} A = \mathbb{C}[x], & \lambda \neq 0 \\ B, & \lambda = 0 \end{cases}$

<u>Suppose $\lambda \neq 0$.</u> If V is a simple module over $B = \mathbb{C}[x] \rtimes S_2$ then

$$V \cong ({}^{1}S)^{\oplus \ell_S} \oplus ({}^{\sigma}S)^{\oplus \ell_S}$$

for any simple $\mathbb{C}[x]$-submodule of B.

and $V \cong \mathrm{Ind}_{\mathbb{C}[x]}^{B} V(S)$.

$\cong \mathbb{C}^\ell$

$x \longmapsto \lambda \mathrm{Id}_{\ell \times \ell}$