HW discussion

$A\langle X\rangle$ revisited

$A \longrightarrow A\langle X\rangle$ gives a pair of adjoint functors:

$\mathrm{Res}_A^{A\langle X\rangle} : {}_{A\langle X\rangle}\underline{\mathrm{Mod}} \to {}_A\underline{\mathrm{Mod}}$

$\mathrm{Ind}_A^{A\langle X\rangle} : {}_A\underline{\mathrm{Mod}} \to {}_{A\langle X\rangle}\underline{\mathrm{Mod}}$

what we called the forgetful functor $\mathcal{O}$.

$$\mathrm{Ind}_A^{A\langle X\rangle} M = \boxed{A\langle X\rangle \otimes_A M}$$

$$A\langle X\rangle = \mathrm{Span}\left\{ a_1 X a_2 X \cdots X a_n \,\middle|\, \begin{matrix} n \geq 0 \\ a_i \in A \end{matrix} \right\}$$

$$\Rightarrow A\langle X\rangle \otimes_A M = \mathrm{Span}\{ a_1 X a_2 X \cdots X a_n \otimes m \mid a_i \in A,\ m \in M \} =$$

$$= \mathrm{span}\left\{ a_1 X a_2 X \cdots a_{n-1} X \otimes m \,\middle|\, \begin{matrix} a_i \in A \\ m \in M \end{matrix} \right\}$$

More precisely, using

$A\langle X\rangle = A \oplus (A\langle X\rangle \cdot XA)$ we get

$$\Rightarrow A\langle X\rangle \otimes_A M = M \oplus (A\langle X\rangle X \otimes_{\Bbbk} M)$$

Morita equivalence for skew group algebras.

① General setting:

A algebra , $e \in A$ idempotent ($e^2 = e$)

Try to construct Morita context:

$$(A,\ eAe,\ eA,\ Ae,\ \tau,\ \mu)$$

eAe: algebra with $1_{eAe} = e$

$$\tau : eA \otimes_A Ae \longrightarrow eAe$$
$$ea \otimes be \longmapsto eabe$$

$$\mu : Ae \otimes_{eAe} eA \longrightarrow A$$
$$ae \otimes eb \longmapsto aeb$$

τ is clearly surjective: $\tau(ea \otimes e) = eae$

μ is surjective iff $A = AeA$

$\iff 1_A = \sum_i a_i e b_i$ for some $a_i, b_i \in A$

The τ, μ associativity conditions follow from associativity in A.

<u>Thm</u> (Actually part of Morita I)
If μ, τ are surjective then they are isomorphisms.

$\Longrightarrow$ If $\underline{\underline{AeA = A}}$, then the above is a Morita context, hence
$A \overset{\text{Mor.}}{\approx} eAe$.

<u>Ex.</u> $A = M_n(\mathbb{k})$, $e = \left(\begin{array}{c|c} I_k & 0 \\ \hline 0 & 0 \end{array}\right) = \sum_{i=1}^{k} E_{ii}$

$E_{ij} = E_{i1} \cdot e \cdot E_{1j}$

$\Rightarrow M_n(\mathbb{k}) = M_n(\mathbb{k}) \cdot e \cdot M_n(\mathbb{k})$

$\Longrightarrow M_n(\mathbb{k}) \overset{\text{Mor.}}{\approx} e M_n(\mathbb{k}) e = \left(\begin{array}{c|c} M_k(\mathbb{k}) & 0 \\ \hline 0 & 0 \end{array}\right)$

$\cong M_k(\mathbb{k})$

② Skew group alg setting:

R comm. alg

G finite group, $|G| \in \mathbb{k}^\times$ acting faithfully on R:

$G \hookrightarrow \mathrm{Aut}_{\mathbb{k}}(R)$.

$$A = R \rtimes G = \left\{ \sum_{g \in G} r_g \cdot g \;\middle|\; r_g \in R \right\}$$

$$g \cdot r = g(r)\, g \quad \forall g \in G,\ r \in R.$$

$$e = \frac{1}{|G|} \sum_{g \in G} g \in \Bbbk G \subset A$$

Check $e^2 = e$ (Exercise).

Question: 1) When is $AeA = A$?

2) What is eAe?

2) $e\, r \cdot g\, e = e r e = \frac{1}{|G|} \sum_{g \in G} g(r) g e =$

Check: $g \cdot e = e = e \cdot g \quad \forall g \in G$

$$= \frac{1}{|G|} \sum_{g \in G} g(r)\, e \in R^G e$$

$$R^G = \{ r \in R \mid g(r) = r \ \forall g \in G \}$$

In fact $eAe = R^G e \cong R^G$

as algebras

Thus $R^G \overset{\text{Mor.}}{\approx} R \rtimes G$, provided $AeA = A$.

TFAE i) $AeA = A$ $\qquad A = R \rtimes G$, e as above

ii) $\exists x_i, y_i \in R : \sum_i x_i \cdot g(y_i) = \delta_{g 1_G}$ for all $g \in G$

iii) $R \rtimes G \cong \mathrm{End}_{R^G}(R)$

iv) $R \otimes_{R^G} R \longrightarrow \{\text{Functions } f: G \to R \text{ w/ pointwise operations}\}$

$r \otimes s \longmapsto (g \mapsto r g(s))$

is an isomorphism of algebras.

Problem Show i) $\Leftrightarrow$ ii)

Def Call $R^G \subset R$ Galois extension w/ Galois group G if the above hold.

Check (Problem) $R = \mathbb{C}[x, x^{-1}]$

$G = S_2 \qquad (12).x = -x$

Show that $R^G \subset R$ is a Galois extension.