Morita contexts & Morita equivalence.

<u>Def</u> A Morita context is a 6-tuple $(A, A', M, M', \tau, \mu)$ where

- A and A' are algebras over $\Bbbk$.
- $M = {}_{A'}M_A$, $M' = {}_A M_{A'}$
- $\tau : M' \otimes_{A'} M \longrightarrow A$ is an isomorphism of (A,A)-modules
- $\mu : M \otimes_A M' \longrightarrow A'$ isomorphism of (A', A')-bimodules

Such that
(1) $\mu(x \otimes y') \cdot y = x \cdot \tau(y' \otimes y)$ $\forall x, y \in M$
(2) $x' \cdot \mu(x \otimes y') = \tau(x' \otimes x) \cdot y'$ $\forall x', y' \in M'$

$(1) \wedge (2) \iff$ "$\begin{pmatrix} A & M' \\ M & A' \end{pmatrix}$ is associative"

$$\begin{pmatrix} A & M' \\ M & A' \end{pmatrix} \stackrel{\text{def}}{=} \left\{ \begin{pmatrix} a & x' \\ x & a' \end{pmatrix} \,\middle|\, \begin{array}{l} a \in A,\ a' \in A' \\ x \in M,\ x' \in M' \end{array} \right\}$$

with multiplication

$$\begin{pmatrix} a & x' \\ x & a' \end{pmatrix} \cdot \begin{pmatrix} b & y' \\ y & b' \end{pmatrix} = \begin{pmatrix} ab + \tau(x' \otimes y) & a \cdot y' + x' \cdot b' \\ x \cdot b + a' \cdot y & \mu(x \otimes y') + a'b' \end{pmatrix}$$

[**Problem**: Prove $\cdot$ is associative iff $(1) \wedge (2)$ hold.]

$\Longrightarrow$ Any Morita context gives rise to an algebra $\begin{pmatrix} A & M' \\ M & A' \end{pmatrix}$.

the Morita algebra

Example. $\left(M_n(\mathbb{k}), \mathbb{k}, (\mathbb{k}^n)^* = [\mathbb{k} \cdots \mathbb{k}], \mathbb{k}^n = \begin{bmatrix} \mathbb{k} \\ \vdots \\ \mathbb{k} \end{bmatrix}, \tau, \mu \right)$

$\quad\quad A \quad\quad A' \quad\quad {}_{A'}M_A \quad\quad {}_{A}M'_{A'}$

$$\tau : \begin{bmatrix} \mathbb{k} \\ \vdots \\ \mathbb{k} \end{bmatrix} \underset{\mathbb{k}'}{\otimes} [\mathbb{k} \cdots \mathbb{k}] \longrightarrow M_n(\mathbb{k})$$

both sides dim n^2

$$x \otimes y \longmapsto x \cdot y$$

$$\mu : [\mathbb{k} \cdots \mathbb{k}] \underset{M_n(\mathbb{k})}{\otimes} \begin{bmatrix} \mathbb{k} \\ \vdots \\ \mathbb{k} \end{bmatrix} \longrightarrow \mathbb{k}$$

$$x \otimes y \longmapsto xy$$

Claim $[1 \; 0 \cdots 0] \otimes \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ spans LHS over $\mathbb{k}$

Pf $e_i^T \otimes e_j = \cdots\cdots$ Exercise.

(1), (2) hold by associativity of matrix multiplication.

<u>Thm</u> (Morita I) If $(A, A', M, M', \tau, \mu)$ is a Morita context, then the functors

(i) $M \underset{A}{\otimes} - : {}_A\underline{\mathrm{Mod}} \longrightarrow {}_{A'}\underline{\mathrm{Mod}}$ and

$M' \underset{A'}{\otimes} - : {}_{A'}\underline{\mathrm{Mod}} \longrightarrow {}_A\underline{\mathrm{Mod}}$

define an equivalence of categories:

$${}_A\underline{\mathrm{Mod}} \simeq {}_{A'}\underline{\mathrm{Mod}}$$

(ii) Similarly $- \underset{A}{\otimes} M'$ & $- \underset{A'}{\otimes} M$ give $\underline{\mathrm{Mod}}_A \simeq \underline{\mathrm{Mod}}_{A'}$

<u>Ex</u>. $A = M_n(\mathbb{k})$, $A' = \mathbb{k}$

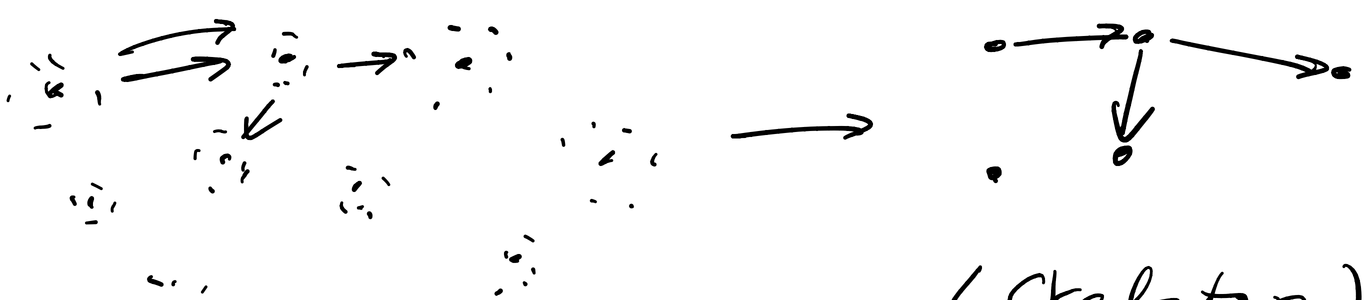
Existence of Morita context we saw, imply

$${}_{M_n(\mathbb{k})}\underline{Mod} \simeq {}_{\mathbb{k}}\underline{Mod} = \underline{Vect}_{\mathbb{k}}$$

category of vector spaces.

In particular, $M_n(\mathbb{k})$ has a <u>unique simple module</u> (up to isomorphism)

Intuition for $\mathcal{C} \simeq \mathcal{C}'$ <u>Ex</u>:



(Skeleton)

Thm (Morita II)

If A and A' are algebras$/\mathbb{k}$ such that ${}_A\underline{\mathrm{Mod}} \simeq {}_{A'}\underline{\mathrm{Mod}}$ then

(i) there exists a Morita context $(A, A', M, M', \tau, \mu)$.

(ii) If F, G are functors realizing the equivalence, then

$$F \approx - \otimes_A M' \quad , \quad G \approx - \otimes_{A'} M$$

natural isomorphism

Def Two algebras A, A' over $\Bbbk$ are Morita equivalent if one of the equivalent conds hold:

1) ${}_A\underline{Mod} \simeq {}_{A'}\underline{Mod}$

2) $\exists$ Morita context $(A, A', M, M', \tau, \mu)$

3) $\underline{Mod}_A \simeq \underline{Mod}_{A'}$

Notation $A \overset{Mor.}{\simeq} A'$