RECAP

$R \xrightarrow{f} A$ morphism of algebras.

$$
\begin{tikzcd}
{}_R\underline{Mod} \arrow[r, shift left, "\mathrm{Ind}_R^A"] \arrow[r, phantom, "\perp"] & {}_A\underline{Mod} \arrow[l, shift left, "\mathrm{Res}_R^A"]
\end{tikzcd}
$$

$a \cdot r = a f(r)$

$$\mathrm{Ind}_R^A(M) = A \underset{R}{\otimes} M \qquad A \otimes N \longrightarrow N$$
$$a \otimes n \longmapsto an$$

$$\mathrm{Res}_R^A(N) = \mathrm{Hom}_{{}_A\underline{Mod}}(A, N) \cong N \text{ as } R\text{-modules}$$

$$\phi \longmapsto \phi(1_A)$$
$$(a \longmapsto an) \longleftarrow n$$

Examples. R algebra, $\sigma \in \mathrm{Aut}_{k\text{-Alg}}(R)$,
$\delta : R \to R$ σ-derivation,
$A = R[t; \sigma, \delta]$ skew pol ring (Ore extension)
$R \xrightarrow{f} A$, $f(r) = r1_A$ (const. pol.)

Specialize to: $R = k[x]$, $\sigma = \mathrm{Id}_R$, δ the unique (Id-) derivation determined by $\delta(x) = 1$ ($\delta = \frac{d}{dx}$). Then $A = k[x][t; \mathrm{Id}, \frac{d}{dx}] \cong A_1(k)$ Weyl alg.

Let $\alpha \in k$. Let $m_\alpha = (x - \alpha)$ be the corresponding maximal ideal of $k[x]$.
Let $k_\alpha = k[x]/m_\alpha$ regarded as left $k[x]$-module.

Thus $k_\alpha = k \cdot 1_\alpha$, where $1_\alpha := 1 + (x-\alpha) \in \frac{k[x]}{(x-\alpha)}$

and $p(x) \cdot 1_\alpha = p(\alpha) 1_\alpha \quad \forall p(x) \in k[x]$.

Goal: Describe $\mathrm{Ind}_R^A(k_\alpha)$.

$$\mathrm{Ind}_R^A(k_\alpha) = A \otimes_R k_\alpha = k[x][t; \mathrm{Id}, \tfrac{d}{dx}] \otimes_{k[x]} k_\alpha =$$

$= \{$ Recall: Skew pol rings are free as left R-modules:

$$A = \bigoplus_{n=0}^{\infty} R t^n = \bigoplus_{n=0}^{\infty} t^n \cdot R$$

$$\Rightarrow A \otimes_R \overset{\text{left } R\text{-mod}}{V} \cong \bigoplus_{n=0}^{\infty} (t^n \cdot k \otimes_k V) \quad \}$$

More detail: $A = \bigoplus_{n=0}^{\infty} t^n \cdot R$

Let $T = \bigoplus_{n=0}^{\infty} k t^n$ k-vector space.

Consider $T \times R \longrightarrow A$

$(t^n, r) \longmapsto t^n \cdot r$ $\forall n \geq 0,\ r \in R$

k-bilinear $\Longrightarrow$ get linear map

$T \otimes_k R \longrightarrow A$, $t^n \otimes r \longmapsto t^n r$

Conversely, $\exists$ linear map $A \longrightarrow T \otimes_k R$

$t^n r \longmapsto t^n \otimes r$

In fact, $T \otimes_k R \cong A$ as right R-modules.

$$\Longrightarrow\quad A \underset{R}{\otimes} k_\alpha \cong \left(T \underset{k}{\otimes} R\right) \underset{R}{\otimes} k_\alpha$$

$$\cong T \underset{k}{\otimes} \left(R \underset{R}{\otimes} k_\alpha\right)$$

$$r \otimes 1_\alpha \longmapsto (r 1_\alpha)$$

$$\cong T \underset{k}{\otimes} k_\alpha \cong T$$

as k-vector spaces

$$t^n \otimes 1_\alpha \longmapsto t^n$$

$$\mathrm{Ind}_R^A(k_\alpha)$$

$$\cong \underbrace{T \underset{k}{\otimes} k_\alpha}_{} = \left(\bigoplus_{n=0}^{\infty} k t^n \underset{k}{\otimes} k_\alpha\right) \cong$$

$$\cong \underbrace{\bigoplus_{n=0}^{\infty} k \cdot (t^n \otimes 1_\alpha)}_{}$$

Action of generators of A on basis $\{t^n \otimes 1_\alpha\}_{n=0}^{\infty}$?

A is generated by $\{x, t\}$ as k-alg.

$$t \cdot (t^n \otimes 1_\alpha) = t^{n+1} \otimes 1_\alpha$$

$$x \cdot (t^n \otimes 1_\alpha) = (xt^n) \otimes 1_\alpha \in A \otimes_R k_\alpha$$

$$= \left(t^n x + [x, t^n]\right) \otimes 1_\alpha$$

$[x,t] = -1$
since $tx - xt = 1$

$$= (t^n x - n t^{n-1}) \otimes 1_\alpha$$

$$= t^n x \otimes 1_\alpha - n t^{n-1} \otimes 1_\alpha$$

$$= t^n \otimes x . 1_\alpha - n t^{n-1} \otimes 1_\alpha$$

$$= \alpha\, t^n \otimes 1_\alpha - n t^{n-1} \otimes 1_\alpha$$

$\cancel{\alpha = 0}$: Can think of this as $k[t]$ with action $t \mapsto$ mul. by t, $x \mapsto -d/dt + \alpha \mathrm{Id}_{k[t]}$ $\blacksquare$

GWAs. $A = R(\sigma, t)$.

$$R \longrightarrow A \quad , \quad r \longmapsto r 1_A$$

As right R-module, $A = \left(\bigoplus_{n=1}^{\infty} Y^n R\right) \oplus 1_A R \oplus \left(\bigoplus_{n=1}^{\infty} X^n \cdot R\right)$

i.e. A is free, with basis $\{Y^n\}_{n=1}^{\infty}, \{1_A\}, \{X^n\}_{n=1}^{\infty}$
as right R-module

M left R-module, (eg. R/m m maximal left ideal).

as before

$$\widehat{M} = \mathrm{Ind}_R^A M = A \underset{R}{\otimes} M = \bigoplus_{n \in \mathbb{Z}} \left(k Z^{(n)} \underset{R}{\otimes} M\right)$$

where $Z^{(n)} = \begin{cases} X^n & n > 0 \\ 1 & n = 0 \\ Y^{|n|} & n < 0 \end{cases}$

Problem Suppose R is an integral domain$/k$ and $t \neq 0$. Prove that any simple left A-module V, containing a weight vector for R (i.e. $v \in V, v \neq 0, \mathfrak{m} v = 0$ for some maximal ideal $\mathfrak{m}$ of R) is a quotient of an induced module $\widehat{M}$ for some M.

GWA Weyl alg

$$\mathbb{C}[t](\sigma: t \mapsto t-1, t) \longrightarrow \mathbb{C}\langle x, \partial / [\partial, x] = 1\rangle$$

$$X \longmapsto x$$

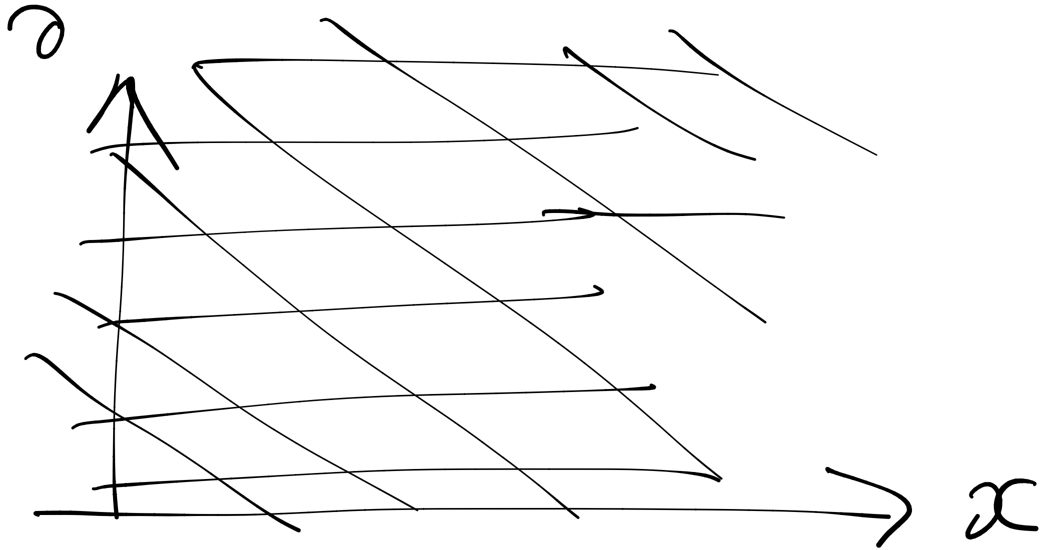
$$Y \longmapsto \partial$$

$$t \longrightarrow \partial x$$

$\{\partial^n x^m\}_{n,m=0}^{\infty}$ basis for $A_1(k)$

$$A_1(k) = \bigoplus_{n=0}^{\infty} \partial^n \cdot k[x] = \left(\bigoplus_{n=1}^{\infty} \partial^n k[\partial x]\right) \oplus k[\partial x] \oplus \left(\bigoplus_{n=1}^{\infty} x^n k[\partial x]\right)$$

Skew pol alg. GWAs



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