Tensor Products II

A, B, C $\Bbbk$-algebras ($\Bbbk$ comm. ring eg. $\mathbb{Z}, \mathbb{C}$)

$X = {}_A X_B$ (A,B)-bimodule $\in {}_A\underline{\mathrm{Mod}}_B$ cat of (A,B)-bimodules

$Y = {}_B Y_C$ (B,C)-bimodule

$Z = {}_A Z_C$ (A,C)-bimodule

Consider maps $f: X \times Y \longrightarrow Z$ such that

1) $f(a_1x_1 + a_2x_2, y) = a_1 f(x_1, y) + a_2 f(x_2, y)$

2) $f(x, y_1c_1 + y_2c_2) = f(x, y_1)c_1 + f(x, y_2)c_2$

3) $f(xb, y) = f(x, by)$

B-balancing cond

Let $\mathrm{Bal}(X \times Y, Z)$ denote the $\Bbbk$-module of all such maps.

$$(afc)(x,y) = af(x,y)c$$
$$= f(ax, yc)$$

$$(afc)(a_1 x, y) = f(a \underset{\sim}{a_1} x, yc)$$

$\Rightarrow \mathrm{Bal}(X \times Y, Z)$ is not an (A,C)-bimod in general

Remark 1) If $f: R \to S$ is morphism of $\Bbbk$-algebras then S is both an (R,S)-bimodule: $r \cdot x \cdot s = f(r) x s$ $\forall r \in R$ $\forall x, s \in S$

and an (S,R)-bimodule: $s \cdot x \cdot r = s x f(r)$

2) ${}_A\underline{\mathrm{Mod}}_{\Bbbk} = {}_A\underline{\mathrm{Mod}}$

If ${}_A Z_C \xrightarrow{g} {}_A Z'_C$ is a morphism in ${}_A\underline{\mathrm{Mod}}_C$ then we get a map

$$\mathrm{Bal}(X \times Y, Z) \longrightarrow \mathrm{Bal}(X \times Y, Z')$$ of $\Bbbk$-modules

So ${}_AX_B$, ${}_BY_C$ gives a covariant functor ${}_A\underline{\mathrm{Mod}}_C \longrightarrow \underline{\mathrm{Vect}}_{\mathbb{k}}$ $(= {}_{\mathbb{k}}\underline{\mathrm{Mod}})$

$$Z \longmapsto \mathrm{Bal}(X \times Y, Z)$$

$\underline{\text{Def}}$ The tensor product of ${}_AX_B$ & ${}_BY_C$ is an object representing the above functor: $X \underset{B}{\otimes} Y \in {}_A\underline{\mathrm{Mod}}_C$

$$\mathrm{Hom}_{{}_A\underline{\mathrm{Mod}}_C}(X \underset{B}{\otimes} Y, Z) \cong \mathrm{Bal}(X \times Y, Z)$$

($\Rightarrow X \underset{B}{\otimes} Y$ is unique up to a unique isom., if exists)

Existence: $X \underset{B}{\otimes} Y \overset{\text{def}}{=} \dfrac{F_{{}_A\mathrm{Mod}_C}(X \times Y)}{\left\langle \begin{array}{l} (xb, y) - (x, by) \\ (a_1x_1 + a_2x_2, y) - \cdots \\ (x, y_1c_1 + y_2c_2) - \cdots \end{array} \right\rangle}$

$\bigoplus_{(x,y) \in X \times Y} A \cdot (x,y) \cdot C$

Notation: $x \otimes y$ = Image of (x,y) in $X \underset{B}{\otimes} Y$

$\Rightarrow \quad X \otimes_B Y = \{$ sums of $x \otimes y$ subject to $(x,y) \longmapsto x \otimes y$ belonging to $\mathrm{Bal}(X \times Y, X \otimes Y)\}$

<u>Note</u>:

$$\begin{tikzcd}
X \times Y \arrow[r, "\otimes"] \arrow[dr, "\forall f \text{ Balanced}"'] & X \otimes Y \arrow[d, dashed, "\exists!\ (A,C)\text{-bimod map.}"] \\
 & Z
\end{tikzcd}
\quad , (x,y) \mapsto x \otimes y$$

Ex $\quad A = B = C = \mathbb{Z} \quad k = \mathbb{Z}$

Claim: $\mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/3\mathbb{Z} = 0$

Pf: $1 \otimes 1 = 1 \cdot 3 \otimes 1 = 1 \otimes 3 \cdot 1 = 1 \otimes 0 = (1 \otimes 0) \cdot 0 = 0$

Properties 1) $A = B$, $X = B$: $\quad {}_B B_B \underset{B}{\otimes} {}_B Y_C \cong {}_B Y_C$

similarly on the right

$B = C$, $Y = B$: $\quad {}_A X_B \underset{B}{\otimes} {}_B B_B \cong {}_A X_B$

2) $$ {}_A X_B \underset{B}{\otimes} \left({}_B Y_C \underset{C}{\otimes} {}_C Z_D \right) \cong \left({}_A X_B \underset{B}{\otimes} {}_B Y_C \right) \underset{C}{\otimes} {}_C Z_D $$

as (A, D)-bimodules

__Theorem__

$$\mathrm{Bal}(X\times Y, Z)\cong \mathrm{Hom}_{{}_B\underline{\mathrm{Mod}}_C}\big(Y, \overbrace{\mathrm{Hom}_{{}_A\underline{\mathrm{Mod}}}(X,Z)}^{\in {}_B\underline{\mathrm{Mod}}_C}\big)$$

Pr__oof__ $\varphi\in \mathrm{Hom}_{{}_A\underline{\mathrm{Mod}}}(X, Z)$, $b\in B$, $c\in C$

$$(b\varphi c)(x) = \varphi(xb)c \Rightarrow (B,C)\text{-module}$$

Given $\Phi: Y\to \mathrm{Hom}_{{}_A\underline{\mathrm{Mod}}}(X,Z)$ from RHS, define $\varphi: X\times Y \to Z$ by $\varphi(x,y) = \big(\Phi(y)\big)(x)$.

$$\begin{aligned}\Rightarrow \varphi(xb, y) = \big(\Phi(y)\big)(xb) &= \big(b\cdot\Phi(y)\big)(x)\\ &= \big(\Phi(by)\big)(x)\\ &= \varphi(x, by)\end{aligned}$$

$$\varphi(ax,y) = \Phi(y)(ax) = a\big(\Phi(y)\big)(x) = a\varphi(x,y)$$

etc $\Rightarrow$ map LHS $\longleftarrow$ RHS.

Conversely, given $\varphi \in \mathrm{Bal}(X \times Y, Z)$, define $\Phi : Y \longrightarrow \mathrm{Hom}_A(X, Z)$ by

$$(\Phi(y))(x) = \varphi(x, y)$$

Can check that $\Phi(y) \in \mathrm{Hom}_{A\text{Mod}}(X, Z)$ & $y \mapsto \Phi(y)$ is in RHS.

Constructions are inverses to each other.

$\blacksquare$

Combining these we get

$$\mathrm{Hom}_{{}_A\underline{\mathrm{Mod}}_C}(X \otimes_B Y, Z) \cong \mathrm{Hom}_{{}_B\underline{\mathrm{Mod}}_C}(Y, \mathrm{Hom}_{{}_A\underline{\mathrm{Mod}}}(X, Z))$$

General form of <u>$\otimes$-Hom adjunction</u>

That is, given $\boxed{X \in {}_A\underline{\mathrm{Mod}}_B}$ we get:

$$\boxed{\begin{tikzcd} {}_B\underline{\mathrm{Mod}}_C \arrow[r, shift left=2, "X \otimes_B -"] \arrow[r, phantom, "\perp"] & {}_A\underline{\mathrm{Mod}}_C \arrow[l, shift left=2, "\mathrm{Hom}_{{}_A\underline{\mathrm{Mod}}}(X, -)"] \end{tikzcd}}$$

OFTEN: $C = \Bbbk$

Application: $H \leq G \implies \Bbbk H \hookrightarrow \Bbbk G$

$\implies \Bbbk G \quad (\Bbbk G, \Bbbk H)$-bimodule.

Suppose Y is a left $\Bbbk H$-module.

$= (\Bbbk H, \Bbbk)$-bimodule.

$A = \Bbbk G, \; B = \Bbbk H, \; C = \Bbbk$

$X = \Bbbk G$

$\implies \Bbbk G \otimes_{\Bbbk H} Y$ is a left $\Bbbk G$-module.

$$\Bbbk G \otimes_{\Bbbk H} - : {}_{\Bbbk H}\underline{\mathrm{Mod}} \longrightarrow {}_{\Bbbk G}\underline{\mathrm{Mod}}$$

$\mathbb{k}G$-module

$$\mathrm{Hom}_{\mathbb{k}G}\left(\mathbb{k}G \otimes_{\mathbb{k}H} Y,\ Z\right) \cong$$

$\mathrm{Ind}_{\mathbb{k}H}^{\mathbb{k}G}\,Y$

$$\cong \mathrm{Hom}_{\mathbb{k}H}\Big(Y,\ \underbrace{\mathrm{Hom}_{\mathbb{k}G}(\mathbb{k}G, Z)}_{\cong\, \mathrm{Res}_{\mathbb{k}H}^{\mathbb{k}G} Z}\Big)$$

Frobenius Reciprocity

Problem: Show that

$$A/I \otimes_A M \cong M/IM \text{ as left } A\text{-modules}$$

where A $\mathbb{k}$-alg, $I \subseteq A$ two-sided ideal

& M left A-module.

(=> A/I (A,A)-bimodule

& M $(A,\mathbb{k})$-bimodule)

Future:

$$A^G \longrightarrow A$$

$$\begin{array}{c} A^G \\ \| \\ \{a \in A \mid g(a) = a \ \forall g \in G\}. \end{array}$$

$$A^G \overset{\text{Mor.}}{\approx} A \# G$$