Tensor products.

V, W vector spaces$/k$

Pick bases $\{v_i\}_{i\in I}$, $\{w_j\}_{j\in J}$ for V, W

Def (First version) The tensor product $V \otimes W = V \otimes_k W$ is defined as the vector space with basis $\{v_i \otimes w_j\}_{(i,j)\in I\times J}$.

If $v = \sum_i \lambda_i v_i \in V$, $w = \sum_j \mu_j w_j \in W$

define $v \otimes w = \sum_{(i,j)\in I\times J} \lambda_i \mu_j \cdot v_i \otimes w_j$

__Note__ 1) The map

$$\alpha : V \times W \longrightarrow V \otimes W$$
$$(v, w) \longmapsto v \otimes w$$

is a bilinear map. (Check!)

2) If U is any vector space and $\beta : V \times W \to U$ is a bilinear map then we get a linear map

$$\bar{\beta} : V \otimes W \longrightarrow U$$

given by $\bar{\beta}(v_i \otimes w_j) = \beta(v_i, w_j)$.

&

$$\begin{tikzcd}
V \times W \arrow[r, "\alpha"] \arrow[dr, bend right, "\beta"'] & V \otimes W \arrow[d, dashed, "\overline{\beta}"] \\
& U
\end{tikzcd}$$

commutes: $\beta(v_i, w_j) = \overline{\beta}(v_i \otimes w_j)$

$= \overline{\beta} \circ \alpha(v_i, w_j)$.

$$\mathrm{Bil}(V \times W, U) \longrightarrow \mathrm{Hom}_k(V \otimes W, U)$$

$$\beta \longmapsto \overline{\beta}$$

Def (second version) Given vector spaces V, W, their tensor product is a pair $(V \otimes W, \alpha)$ where $V \otimes W$ is a v.sp. and α is a bilinear map $\alpha : V \times W \to V \otimes W$ **such that** given any pair (U, β) where U is a v.sp. & $\beta : V \times W \to U$ is a bil. map $\exists !$ linear map $\overline{\beta} : V \otimes W \to U$ s.t.

$$\begin{tikzcd} V \times W \arrow[r, "\alpha"] \arrow[dr, "\beta"'] & V \otimes W \arrow[d, "\overline{\beta}"] \\ & U \end{tikzcd}$$

commutes.

__Note__ Fix V, W consider

$\mathcal{B}(V,W)$ = category with

objects: pairs (U,β) $\quad$ U v.s.p. $\quad$ $\beta : V\times W \xrightarrow{\text{bil.}} U$

& morphism $\varphi : (U,\beta) \longrightarrow (U',\beta')$ is a commutative diagram:

$$\begin{tikzcd} & V\times W \arrow[dl, "\beta"'] \arrow[dr, "\beta"] & \\ U \arrow[rr, "\varphi"'] & & U' \end{tikzcd}$$

φ linear.

Then of the tensor product $(V\otimes W, \alpha)$ is an initial object in $\mathcal{B}(V,W)$.

$\Rightarrow$ Uniqueness up to unique isomorphism.

Existence: Define

$$V \otimes W = \frac{F}{S}, \quad \alpha : V \times W \to V \otimes W$$
$$(v, w) \longmapsto (v, w) + S$$

F = vector space with basis $V \times W$

$$S = \operatorname{span}_k \left\{ \begin{array}{l} (\lambda_1 v_1 + \lambda_2 v_2, w) - (\lambda_1 (v_1, w) + \lambda_2 (v_2, w)), \\ (v, \lambda_1 w_1 + \lambda_2 w_2) - (\lambda_1 (v, w_1) + \lambda_2 (v, w_2)) \end{array} \;\middle|\; \begin{array}{l} v, v_1, v_2 \in V, \\ w, w_1, w_2 \in W \end{array}, \lambda_1, \lambda_2 \in k \right\}$$

Check: If $\beta : V \times W \longrightarrow U$ is bilinear map, it extends (being a function from a set) to a linear map $\hat{\beta} : F \longrightarrow U$.

β bilinear $\Longrightarrow S \subset \ker \hat{\beta}$

so get induced linear map

$\bar{\beta} : V \otimes W \longrightarrow U$. Can check $\bar{\beta}$ is unique s.t.

$$\begin{tikzcd} V \times W \arrow[r, "\alpha"] \arrow[dr, "\beta"'] & V \otimes W \arrow[d, "\bar{\beta}"] \\ & U \end{tikzcd}$$

commutes.

Ex. A an algebra with multiplication map $m : A \times A \to A$.

$(a, b) \mapsto ab$

m is bilinear, hence induces a linear map

$$\bar{m} : A \otimes A \longrightarrow A$$

(Conversely, any linear map $A \otimes A \to A$ pulls back via $A \times A \to A \otimes A$ to a bilinear map $A \times A \to A$.)

$$\boxed{\mathrm{Bil}(V \times W, U) \cong \mathrm{Hom}_k(V \otimes W, U)}$$

Ex. $$k^n \oplus k^m \cong k^{n+m}$$
$$k^n \otimes_k k^m \cong k^{n\cdot m}$$
$$\dim(V \otimes_k W) = (\dim V)(\dim W).$$

Adjunction . First:

$$\mathrm{Bil}(V \times W, U) \cong \mathrm{Hom}_k(V, \mathrm{Hom}_k(W, U))$$
$$\Phi \longmapsto (v \mapsto (w \mapsto \Phi(v,w)))$$

$$\Rightarrow \boxed{\mathrm{Hom}_k(V, \mathrm{Hom}_k(W,U)) \cong \mathrm{Hom}_k(V \otimes W, U)}$$

so $-\otimes W$ is left adjoint to $\mathrm{Hom}(W,-)$

$\boxed{(\otimes, \mathrm{Hom})\text{-adjointness}}$

$-\otimes W \dashv \mathrm{Hom}(W,-)$

$$\begin{tikzcd} \underline{\mathrm{Vect}} \arrow[r, bend left, "-\otimes W"] \arrow[r, phantom, "\perp"] & \underline{\mathrm{Vect}} \arrow[l, bend left, "\mathrm{Hom}(W,-)"] \end{tikzcd}$$

$\lambda \otimes v \longmapsto \lambda v$

$k \otimes_k V \cong V$

A k-alg, V an A-module

$A \otimes_k V \longrightarrow V$

$a \otimes v \longmapsto a.v$

$(ab).v = a.(b.v)$, $1.v = v$

Bonus:

A monoid in $(\underline{Set}, \times, *)$ = Monoid

A monoid in $(\underline{Vect}, \otimes_k, k)$ = k-algebra.

A monoid in $(H\text{-Mod}, \otimes_k, k)$ = H-module algebra

H ← Hopf alg