Group-graded algebras.

k field, A k-algebra, $G \ni e =$ identity group

$\{A_g\}_{g\in G}$ a G-gradation of A:

$$A = \bigoplus_{g\in G} A_g \qquad 1_A \in A_e$$

$$A_g A_h \subseteq A_{gh}$$

__Examples.__ 1) $A = kG$ group (ring) algebra

$= \{ \sum_{g\in G} \lambda_g \cdot g \mid \lambda_g \in k, \text{ almost all} = 0 \}$

Multiplication in A is bilinear extension of that in G.

G-gradation: $(kG)_g = k \cdot g$

$1_{kG} = 1_k \cdot e = e \in ke = (kG)_e$

$(kG)_g (kG)_h = \underbrace{(kg)(kh)} = kgh.$

$\{\alpha g \cdot \beta h \mid \alpha, \beta \in k\}$

A_e identity component

2) R any k-algebra

$RG = \left\{ \sum r_g \cdot g \;\middle|\; \begin{array}{l} r_g \in R \\ \text{almost all} = 0 \end{array} \right\}$

group algebra over R.

(= group ring over R)

$(RG)_g = Rg$

- $(r_g g) \cdot (r_h h) := (r_g r_h) \cdot (gh)$
- extend bi-additively

3) Skew group algebras

R k-algebra

$\sigma : G \longrightarrow \mathrm{Aut}_k(R)$ group homomorphism.

$g \mapsto \sigma_g$

Define $R *_\sigma G = R * G = \boxed{R \rtimes_\sigma G} = R \# G$

$= R \# kG$

by

$$R \rtimes_\sigma G = \Big\{ \sum_{g \in G} r_g \, g \;\Big|\; r_g \in R, \text{ a.a.} = 0 \Big\}$$

with mult:

$$(r g)(s h) = \big(r \cdot \sigma_g(s)\big) g h$$

extended bi-additively ($= \mathbb{Z}$-bilinearly)

$$\Rightarrow \left(\sum_{g\in G} r_g\, g\right)\left(\sum_{h\in G} s_h\, h\right) =$$

$$= \sum_{g,h\in G} r_g\, \sigma_g(s_h)\, gh =$$

$$= \sum_{k\in G} \left(\sum_{\substack{(g,h)\in G^2 \\ gh=k}} r_g\, \sigma_g(s_h)\right)\cdot k \in R \rtimes_\sigma G$$

Mult is associative: $rg(sh\,tk) \overset{\sigma \text{ homomorphism}}{=} (rg\,sh)tk$

(CHECK!)

Alternatively: Define

$$R \rtimes_\sigma G = k\left\langle R \cup G \;\middle|\; \begin{array}{l} gr = \sigma_g(r)g \\ gh = \underbrace{g\cdot h}_{\text{in } G} \\ rs = \underbrace{r\cdot s}_{\text{in } R} \\ \text{etc} \end{array} \right\rangle$$

& use Diamond Lemma to prove

$$R \rtimes_\sigma G = \bigoplus_{g \in G} Rg \quad \text{as left } R\text{-modules}$$

$$= \bigoplus_{g \in G} gR \quad \text{as right } R\text{-modules}$$

TFAE

1) $gr = \sigma_g(r)g$ $\forall g \in G,\ r \in R$

2) $(rg)(sh) = (r\sigma_g(s))gh$ $\forall g,h \in G$
$\forall r,s \in R$

1) $\Rightarrow$ 2) : $(rg)(sh) = r\cdot(gs)\cdot h$ (by assoc.)
$\overset{1)}{=} r\cdot(\sigma_g(s)g)\cdot h$
$= (r\sigma_g(s))gh$ (by assoc.)

2) $\Rightarrow$ 1) Take $r = 1_R$, $h = e_G$. $\Rightarrow gs = \sigma_g(s)g$

$\underline{\text{Ex}}$ of skew group alg

$R = \mathbb{C}[x_1, x_2, \ldots, x_n]$ = free comm alg on $\{x_1, \ldots, x_n\}$

$G = S_n$

$\varphi : S_n \longrightarrow \mathrm{Aut}_{\mathbb{C}}(R)$

$\sigma \longmapsto \varphi_\sigma$

$\varphi'_\sigma : \{x_1, x_2, \ldots, x_n\} \longrightarrow R$

$x_i \longmapsto x_{\sigma(i)}$

extends to a $\mathbb{C}$-alg homomorphism

$\varphi_\sigma : R \to R$, $x_i \mapsto x_{\sigma(i)}$.

$\underline{\text{Note}}$ $\varphi_\sigma \varphi_{\sigma^{-1}} = \mathrm{Id}_R$

φ group hom is clear. Get

$$R \rtimes_{\varphi} S_n = \bigoplus_{\sigma \in S_n} R\sigma$$

$$(12)\cdot(x_1 + x_2^2) = (x_2 + x_1^2)\cdot(12)$$

Spoiler: $R \rtimes_{\varphi} S_n$ is "Morita equivalent"

to $R^{S_n} = \{ r \in R \mid \varphi_\sigma(r) = r \ \forall \sigma \in S_n \}$

algebra of invariants = symmetric pol's.

$n = 3$: $x_1^2 x_2 x_3 + x_1 x_2^2 x_3 + x_1 x_2 x_3^2 \in R^{S_n}$.

(Paper: M. Cohen Morita equivalence for skew group rings.).

Crossed products.

R k-alg, G group,

$\sigma : G \longrightarrow \mathrm{Aut}_k(R)$

$\alpha : G \times G \longrightarrow R^{\times} = \{\text{group of units of } R\}$ ($=$ invertible elts)

subject to conditions.

$\{u_g\}_{g \in G}$ set indexed by G

Define a product on

$$R *_{\sigma,\alpha} G = \bigoplus_{g \in G} R \cdot u_g = \left\{ \sum r_g \cdot u_g \mid \cdots \right\}$$

$$\boxed{u_g u_h = \alpha(g,h) u_{gh}}$$

w. mult:

$$(r u_g)(s u_h) = r\, \sigma_g(s)\, \alpha(g,h) u_{gh}$$

We have:

$$u_g(u_h u_k) = u_g \cdot (\alpha(h,k) u_{hk}) =$$
$$= \sigma_g(\alpha(h,k))\, u_g u_{hk} =$$
$$= \sigma_g(\alpha(h,k))\, \alpha(g,hk)\, u_{ghk}$$

$$(u_g u_h) u_k = \alpha(g,h)\, u_{gh} u_k = \alpha(g,h)\,\alpha(gh,k)\, u_{ghk}$$

$$\boxed{\sigma_g\big(\alpha(h,k)\big)\,\alpha(g,hk) = \alpha(g,h)\,\alpha(gh,k)} \quad \forall g,h,k \in G.$$

(σ-twisted cocycle identity)

Theorem TFAE for a G-graded algebra $A = \bigoplus_{g\in G} A_g$.

1) $A \overset{\text{as } G\text{-graded algebras}}{\cong} A_e *_{\sigma,\alpha} G$ for some σ, α

2) $\forall g \in G \ \exists\, u_g \in A_g \cap (A^\times)$.

Pf (Sketch)

Assume 2): $\quad u_g u_h \in A_{gh} \ni u_{gh}$.

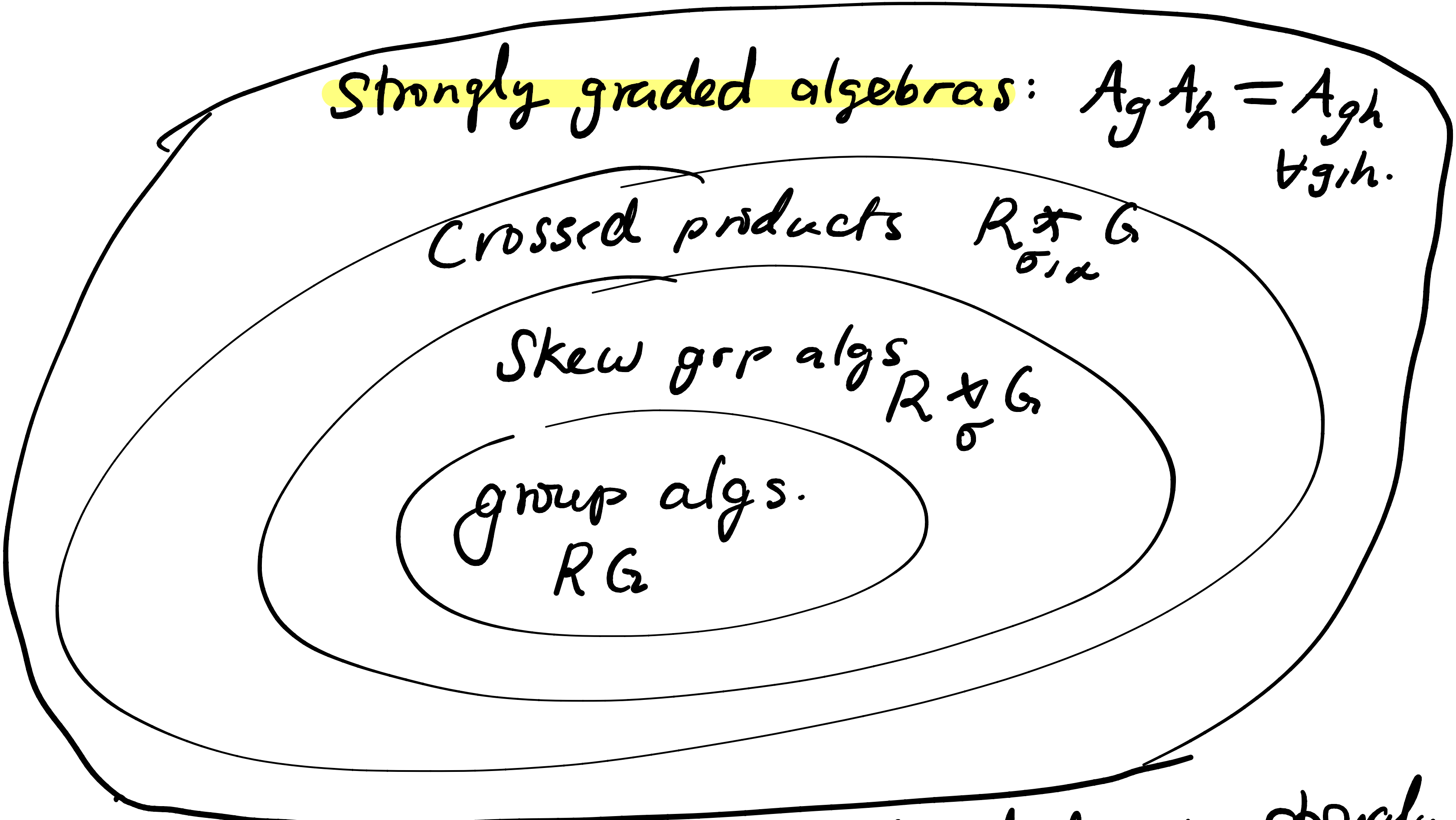
$$u_g u_h (u_{gh})^{-1} \in A_e$$

Put $\quad \alpha(g,h) = u_g u_h (u_{gh})^{-1} \in (A_e)^\times$

Then $\quad u_g u_h = \alpha(g,h)\, u_{gh}$

Problem 1) Finish the proof.

$\blacksquare$



Problem 2): Show that crossed products are strongly graded.