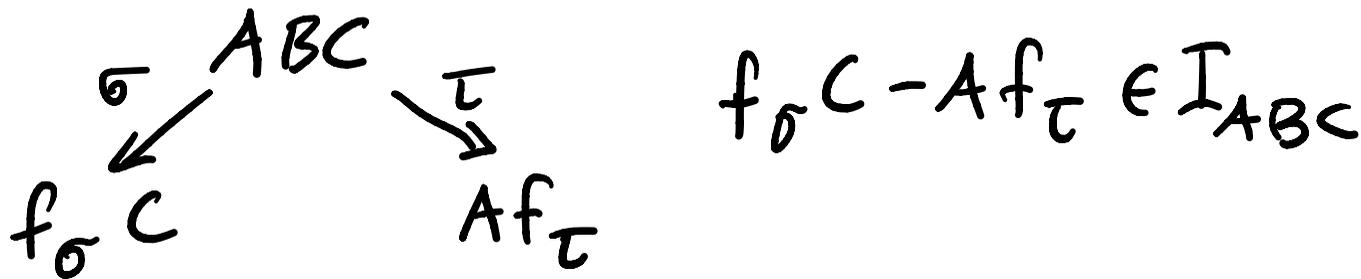


# DIAMOND LEMMA III

$X$   $S \subseteq \langle X \rangle \times k\langle X \rangle$ ,  $\leq$  on  $\langle X \rangle$  compatible with  $S$

$(\sigma, \tau, A, B, C)$  resolvable relative to  $\leq$  if



Note  $I_A = \{ B - r(B) \mid B \in k\langle X \rangle, B < A, r \text{ sequence of reductions} \}$

## Diamond Lemma (concrete version)

If all ambiguities are resolvable relative to  $S$ , then  $\langle X \rangle_{\text{irr}}$  is (mapped to) a  $k$ -basis for  $k\langle X \rangle / I$ .

Application: GWAs.  $R$   $k$ -alg,  $t \in Z(R)$ ,  
 $\sigma \in \text{Aut}_{k\text{-Alg}}(R)$ .  $\Rightarrow A = R(\sigma, t)$  GWA

Let  $B = \{ b_i \}_i$   $k$ -basis for  $R$ .

Then  $b_i b_j = \sum_k c_{ijk} \cdot b_k$ , where  $c_{ijk} \in k$   
&  $t = \sum_i t_i b_i$ ,  $t_i \in k$

$\sigma(b_i) = \sum_j s_{ij} b_j$ ,  $s_{ij} \in k$

Structure constants  
of  $R$  (wrt  $B$ )

$$A = k \langle \{b_i\}_i \cup \{X, Y\} \rangle / \langle \text{Rels.} \rangle$$

Rels:  $X b_i = \sigma(b_i)X = (\sum_j s_{ij} b_j)X$

[Also define  $s_{ij} \in k$  by  $\sigma^{-1}(b_i) = \sum_j s_{ij} b_j$ ]

$$Y b_i = \sigma^{-1}(b_i)Y = (\sum_j s_{ij} b_j)Y$$

$$YX = t = \sum_i t_i b_i$$

$$XY = \sigma(t) = \sum_i t_i \sigma(b_i) =$$

$$b_i b_j = \sum_k c_{ijk} b_k \left. \begin{array}{l} = \sum_i t_i \sum_j s_{ij} b_j \\ = \sum_j (\sum_i t_i s_{ij}) b_j \end{array} \right\}$$

Reduction system:  $\mathcal{X} = \{b_i\}_i \cup \{X, Y\}$

$$S: \begin{cases} X b_i \longrightarrow \sum_j s_{ij} b_j X \\ Y b_i \longrightarrow \sum_j s'_{ij} b_j Y \\ Y X \longrightarrow \sum_i t_i b_i \\ X Y \longrightarrow \sum_j (\sum_i t_i s_{ij}) b_j \\ b_i b_j \longrightarrow \sum_k c_{ijk} b_k \end{cases}$$

DEF:

$$A < B$$

iff

$$1) l(A) < l(B)$$

OR

$$A, B \in \langle \mathcal{X} \rangle$$

$$2) l(A) = l(B) \ \& \ N_z(A) = N_z(B) \ \forall z \in \{X, Y\}$$

$$m(A) = \# \left\{ (z, b_i) \mid \begin{array}{l} z = X \text{ OR } Y \\ b_i \text{ is right} \\ \text{of } z \text{ in } A \end{array} \right\} < m(B)$$

# letters  
in word.

mismatching  
index

Ex  $A = X \xrightarrow{b_i} Y \xrightarrow{b_j} X$   $N_X(A) = 2$   
 $B = b_k \xrightarrow{X} Y \xrightarrow{b_l} X$  etc.  
 $m(A) = 3$  ,  $m(B) = 2$   
 $\Rightarrow B < A$

Can check that  $\leq$  is a semigroup  
 ordering on  $\langle X \rangle$  & that is compatible  
 with  $S$ .

Ambiguities:  $XYb_i$ ,  $YXb_i$ ,  
 $Xb_ib_j$ ,  $Yb_ib_j$   
 $YXY$ ,  $XYX$

$XYb_i$

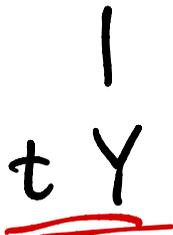
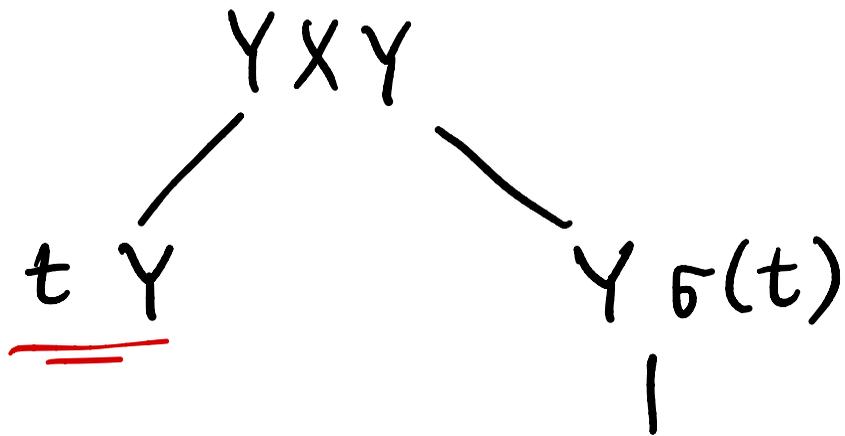
$\sigma(t)b_i$

$X\sigma^{-1}(b_i)Y = b_iXY = b_i\sigma(t) = \underline{\underline{\sigma(t)b_i}}$

$\sum t_j b_j b_i - X \sum_j s_{ij} b_j Y \in I_{XYb_i}$

$$\begin{array}{ccc}
 X b_i b_j & & \sum c_{ijk} b_k \\
 \parallel & & \parallel \\
 \sigma(b_i) X b_j & & X \overbrace{(b_i b_j)} \\
 \parallel & & \parallel \\
 \sigma(b_i) \sigma(b_j) X & = & \sigma(b_i b_j) X \\
 \uparrow & & \\
 s_{ij} & & 
 \end{array}$$

A is an R-ring



Conclusion:

$$\langle X \rangle_{\text{irr}} = \{ b_i t \}_i \cup \{ b_i X^n \}_{i, n \geq 1} \cup \{ b_i Y^n \}_{i, n \geq 1}$$

is a  $k$ -basis for the GWA  $A = R(\sigma(t))$ .

Lie algs. Ex.  $A = \mathcal{O}(x, y, z) \left. \begin{array}{l} xy - yx = z \\ yz - zy = x \\ zx - xz = y \end{array} \right\}$

Put  $[a, b] = ab - ba$

$$zyx$$

$$\begin{aligned} & (yz + [z, y])x \\ &= yzx + [z, y]x \end{aligned}$$

$$\begin{aligned} & z(xy + [y, x]) \\ &= zxy + z[y, x] \end{aligned}$$

$$\begin{aligned} & \downarrow \\ & yxz + y[z, x] + [z, y]x \quad \downarrow \quad xzy + [z, x]y + z[y, x] \end{aligned}$$

$$\begin{aligned} & \downarrow \\ & xyz + [y, x]z + y[z, x] + [z, y]x \quad \downarrow \quad xyz + x[z, y] + [z, x]y + z[y, x] \end{aligned}$$

$$\text{LHS} = \text{RHS} \iff [z, [y, x]] + [[z, x], y] + [x[z, y]] = 0 \quad \underline{\underline{\text{Jacobi}}}$$

Cor.  $\langle X \rangle_{\text{irr}} = \{ x^a y^b z^c \mid a, b, c \geq 0 \}$   
map to a basis in  $A (\cong U(\mathfrak{so}_3(\mathbb{C})))$