DIAMOND LEMMA III

$X \quad S \subseteq \langle X \rangle \times k\langle X \rangle$, $\leq$ on $\langle X \rangle$ compatible with S

(σ, τ, A, B, C) resolvable relative to $\leq$ if

$$\begin{tikzcd}[column sep=small] & ABC \arrow[dl, "\sigma"'] \arrow[dr, "\tau"] & \\ f_\sigma C & & A f_\tau \end{tikzcd} \qquad f_\sigma C - A f_\tau \in I_{ABC}$$

Note $I_A = \{ B - r(B) \mid B \in k\langle X \rangle, B < A,$ r sequence of reductions$\}$

Diamond Lemma (concrete version)

If all ambiguities are resolvable relative to $\leq$, then $\langle X\rangle_{irr}$ is (mapped to) a k-basis for $k\langle X\rangle/I$.

Application: GWAs. R k-alg, $t \in Z(R)$, $\sigma \in \mathrm{Aut}_{k\text{-}Alg}(R)$. $\Longrightarrow A = R(\sigma, t)$ GWA

Let $\mathcal{B} = \{b_i\}_i$ k-basis for R.

Then $b_i b_j = \sum_k c_{ijk} \cdot b_k$, where $c_{ijk} \in k$

& $t = \sum_i t_i b_i$, $t_i \in k$

$\sigma(b_i) = \sum_j s_{ij} b_j$, $s_{ij} \in k$

Structure constants of R (wrt $\mathcal{B}$)

$$A = k\langle \{b_i\}_i \cup \{X, Y\}\rangle / \langle \text{Rels.}\rangle$$

Rels: $X b_i = \sigma(b_i) X = \left(\sum_j s_{ij} b_j\right) X$

[Also define $s'_{ij} \in k$ by $\sigma^{-1}(b_i) = \sum_j s'_{ij} b_j$]

$$Y b_i = \sigma^{-1}(b_i) Y = \left(\sum_j s'_{ij} b_j\right) Y$$

$$YX = t = \sum_i t_i b_i$$

$$XY = \sigma(t) = \sum_i t_i \sigma(b_i) =$$

$$= \sum_i t_i \sum_j s_{ij} b_j$$

$$= \sum_j \left(\sum_i t_i s_{ij}\right) b_j$$

$$b_i b_j = \sum_k c_{ijk} b_k$$

Reduction system: $\mathcal{X} = \{b_i\}_i \cup \{X, Y\}$

$$S: \begin{cases} X b_i \longrightarrow \sum_j s_{ij} b_j X \\ Y b_i \longrightarrow \sum_j s'_{ij} b_j Y \\ YX \longrightarrow \sum_i t_i b_i \\ XY \longrightarrow \sum_j \left(\sum_i t_i s_{ij}\right) b_j \\ b_i b_j \longrightarrow \sum_k c_{ijk} b_k \end{cases}$$

DEF:

$A < B$ iff 1) $\ell(A) < \ell(B)$ OR

$A, B \in \langle \mathcal{X} \rangle$ 2) $\ell(A) = \ell(B)$ & $N_Z(A) = N_Z(B)$ $\forall Z \in \{X, Y\}$

\# letters in word.

misordering index $m(A) = \#\left\{ (Z, b_i) \,\middle|\, \begin{array}{l} Z = X \text{ or } Y \\ b_i \text{ is right} \\ \text{of } Z \text{ in } A \end{array} \right\} < m(B)$

Ex $\quad A = X\, b_i\, Y\, b_j\, X \qquad N_X(A) = 2$ etc.

$\quad B = b_k\, X\, Y\, b_\ell\, X$

$m(A) = 3 \quad , \quad m(B) = 2$

$\Rightarrow B < A$

Can check that $\leq$ is a semigroup ordering on $\langle X \rangle$ & that is compatible with S.

Ambiguities: XYb_i , YXb_i ,

Xb_ib_j , Yb_ib_j

YXY , XYX

$$
\begin{tikzcd}
 & XYb_i \arrow[dl] \arrow[dr] & \\
\sigma(t)b_i & & X\sigma^{-1}(b_i)Y
\end{tikzcd}
$$

$$X\sigma^{-1}(b_i)Y = b_iXY = b_i\sigma(t) = \sigma(t)b_i$$

$$\sigma(t)b_i = \sum t_j b_j b_i \qquad X\sigma^{-1}(b_i)Y = X\sum_j s'_{ij} b_j Y$$

$$\sum t_j b_j b_i - X\sum_j s'_{ij} b_j Y \overset{?}{\in} I_{XYb_i}$$

$$\begin{tikzcd}
 & X\, b_i b_j \arrow[dl, equal] \arrow[dr, equal] & \\
\sigma(b_i)\, X\, b_j \arrow[d, equal] & & X\, \overbrace{(b_i b_j)}^{\sum c_{ijk} b_k} \arrow[d, equal] \\
\sigma(b_i)\,\sigma(b_j)\, X \arrow[rr, equal, "s_{ij}"'] & & \sigma(b_i b_j)\, X
\end{tikzcd}$$

$\boxed{\text{$A$ is an R-ring}}$

$$
\begin{tikzcd}
 & YXY \arrow[dl, no head] \arrow[dr, no head] & \\
tY & & Y\sigma(t) \arrow[d, no head] \\
 & & tY
\end{tikzcd}
$$

Conclusion:

$$\langle \mathcal{X} \rangle_{irr} = \{ b_i 1 \}_i \cup \{ b_i X^n \}_{i, n \geq 1} \cup \{ b_i Y^n \}_{i, n \geq 1}$$

is a k-basis for the GWA $A = R(\sigma, t)$. $\blacksquare$

Lie algs.

Put $[a,b]=ab-ba$

Ex. $A = \mathbb{C}\{x,y,z \mid xy-yx=z,\ yz-zy=x,\ zx-xz=y\}$

$$zyx$$

$$(yz+[z,y])x$$
$$= yzx+[z,y]x$$
$$\downarrow$$
$$yxz+y[z,x]+[z,y]x$$
$$\downarrow$$
$$xyz+[y,x]z+y[z,x]+[z,y]x$$

$$z(xy+[y,x])$$
$$= zxy+z[y,x]$$
$$\downarrow$$
$$xzy+[z,x]y+z[y,x]$$
$$\downarrow$$
$$xyz+x[z,y]+[z,x]y+z[y,x]$$

LHS = RHS $\iff$ $[z,[y,x]]+[[z,x],y]+[x,[z,y]]=0$

<u>cor</u>. $\langle X \rangle_{irr} = \{ x^a y^b z^c \mid a, b, c \geq 0 \}$

map to a basis in A $(\cong U(so_3(\mathbb{C})))$