DIAMOND LEMMA II.

$S \subseteq \langle X \rangle \times k\langle X \rangle$

$\sigma \in S \qquad \sigma = (W_\sigma, f_\sigma)$ (THINK: $\sigma : W_\sigma \rightsquigarrow f_\sigma$)

$r_{A\sigma B} : k\langle X \rangle \to k\langle X \rangle$ k-linear map.

$$C \longmapsto \begin{cases} C & , C \neq AW_\sigma B \\ Af_\sigma B & , C = AW_\sigma B \end{cases}$$

Ex. $\sigma = (xy, yx + 1)$

$r_{y\sigma x^2}(y^2xyx^3) = y^2xyx^3$, $r_{y^2\sigma x^3}$

$a \in k\langle X \rangle$ **reduction-finite** if you "can't reduce forever"

<u>Ex</u>. $X = \{x, y\}$

$$\sigma : xy \longrightarrow 1$$
$$\tau : xy \longrightarrow x^2 y^2$$

Then xy is <u>not</u> reduction-finite.

$$\cdots r_{x^2 \tau y^2}\, \underbrace{r_{x \tau y}\, \underbrace{r_{1 \tau 1}(xy)}_{x^2 y^2}}_{x^3 y^3}$$

$a \in k\langle X\rangle$ is reduction-unique if red-finite & every final reductions is the same.

LEM: $R = \{$ reduction-unique elements $\} \subseteq k\langle X\rangle$ is a k-submodule, and

$$r_S : R \longrightarrow k\langle X\rangle_{irr}$$

is a linear map.

PF $a, b \in R$, $\alpha \in k \Rightarrow \alpha a + b$ is reduction-finite. Pick $r = r_n \cdots r_1$ s.t. $r(\alpha a + b) \in k\langle X\rangle_{irr}$. Since $a, b \in R$ $r' \circ r(a) = r_S(a)$ & $r'' \circ r' \circ r(b) = r_S(b)$ for some r', r''. $\Longrightarrow \underbrace{r'' r' r(\alpha a + b)}_{= r(\alpha a + b)} = \alpha r'' r_S(a) + r_S(b) = \alpha r_S(a) + r_S(b)$ ∎

<u>LEM</u> If $a, b, c \in k\langle X\rangle$ is s.t. for every word A, B, C in a, b, c respectively ABC is reduction-unique, then $a \cdot r(b) \cdot c$ is reduction-unique for every seq of reductions r, and

$$r_S(a \cdot r(b) \cdot c) = r_S(abc).$$

<u>PF</u>. WLOG $a = A$, $b = B$, $c = C$ and $r = r_{D\sigma E}$ single reduction.
Then $A\, r_{D\sigma E}(B)\, C = r_{AD\sigma EC}(ABC)$ which is red. unique since ABC is. Hence
$r_S(r_{AD\sigma EC}(ABC)) = r_S(ABC)$. $\blacksquare$

$\leq$ semigroup partial ordering on $\langle X \rangle$

$$A < B \implies CAD < CBD \quad \forall C, D$$

Ex. $A < B$ iff $\ell(A) < \ell(B)$

$xyz \not< \not> yxz$ — length = # letters in A

Ex. $A < B$ iff $\ell(A) < \ell(B)$

OR $\ell(A) = \ell(B)$ &

$A < B$ in lexicographical order (wrt fixed total order on X).

$xyz < yxz$

$\leq$ compatible with S

if $\forall \sigma \in S$ f_σ is a linear comb. of words $< W_\sigma$.

$$I = \mathrm{Span}_k \left\{ A(W_\sigma - f_\sigma)B \;\middle|\; \begin{array}{l} A, B \in \langle X \rangle \\ \sigma \in S \end{array} \right\}$$

$$I_A = \mathrm{span}_k \left\{ B(W_\sigma - f_\sigma)C \;\middle|\; \begin{array}{l} B, C \in \langle X \rangle \\ \sigma \in S \\ BW_\sigma C < A \end{array} \right\}$$

$$= \mathrm{Span}_k \left\{ B - r_1(B) \;\middle|\; \begin{array}{l} B \in \langle X \rangle,\ B < A \\ r_1 \text{ any reduction} \end{array} \right\}$$

$r_1 = r_{A,\sigma,C_1}$

Ambiguity (σ, τ, A, B, C) is resolvable if

$$
\begin{tikzcd}
& ABC \arrow[dl] \arrow[dr] & \\
f_\sigma C \arrow[dr, dashed] & & A f_\tau \arrow[dl, dashed] \\
& g &
\end{tikzcd}
\qquad
\begin{tikzcd}
& ABC \arrow[dl] \arrow[dr] & \\
A f_\sigma C \arrow[dr, dashed] & & f_\tau \arrow[dl, dashed] \\
& g &
\end{tikzcd}
$$

$(\sigma, \tau, A B, C)$ is resolvable wrt $\leq$ if $f_\sigma C - A f_\tau \in I_{ABC}$ (resp. $A f_\sigma C - f_\tau \in I_{ABC}$).

<u>If</u> $f_\sigma C$ & Af_τ have a common reduction, then $f_\sigma C - Af_\tau$ can be reduced to zero.

$\Rightarrow r_n \cdots r_1(f_\sigma C - Af_\tau) = 0$

Since $\leq$ is compatible with S,

$$f_\sigma < W_\sigma \qquad f_\tau < W_\tau$$

$\Rightarrow f_\sigma < W_\sigma C = ABC$, $Af_\tau < AW_\tau = ABC$

so $f_\sigma C - Af_\tau < ABC \Rightarrow$

$\Rightarrow f_\sigma C - Af_\tau - r_1(f_\sigma C - Af_\tau) \in I_{ABC}$

In fact
$$I_A = \{ B - r(B) \mid B \in \langle X \rangle, B < A, \ r \text{ } \underline{\text{any}} \text{ } \underline{\text{seq}} \text{ of reductions}, \}$$

$$\Rightarrow f_\sigma C - A f_\tau =$$
$$= f_\sigma C - A f_\tau - r_n \cdots r_1 (f_\sigma C - A f_\tau)$$
$$\in I_{ABC} .$$