THE DIAMOND LEMMA.

k assoc. comm. ring w 1

X set

$\langle X \rangle$ free monoid on X = set of words in X

$k\langle X \rangle$ free k-alg on X.

A reduction system is a subset $S \subseteq \langle X \rangle \times k\langle X \rangle$

For $\sigma \in S$ we write $\sigma = (W_\sigma, f_\sigma)$

A reduction is a k-linear map:

$$r_{A\sigma B} : k\langle X \rangle \longrightarrow k\langle X \rangle \quad , \quad A, B \in \langle X \rangle, \ \sigma \in S$$

$$A W_\sigma B \longmapsto A f_\sigma B$$

$$C \longmapsto C \quad , \quad C \in \langle X \rangle, \ C \neq A W_\sigma B$$

- $r_{A\sigma B}$ acts trivially on $a \in k\langle X\rangle$ if the coeff. of $AW_\sigma B$ in a is zero.
- $a \in k\langle X\rangle$ is irreducible (w.r.t. S) if $r_{A\sigma B}$ acts trivially on a $\forall A,B \in \langle X\rangle, \sigma \in S$.

<u>Ex.</u> $X = \{x, y\}$, $S = \{\sigma = (\underbrace{yx}_{W_\sigma}, \underbrace{xy+1}_{f_\sigma})\}$

$$r_{1\sigma 1}(yx) = xy + 1$$

$$r_{x\sigma x^2}(yx^3 + 3\underline{x}y\underline{x}^3 + yx) =$$

$$= \underset{r_{1\sigma x^2}}{yx^3} + 3\,\underline{x(xy+1)x^2} + yx$$

$$xW_\sigma x^2$$

An element a in $k\langle X\rangle$ is irreducible iff it's a lin. comb. of monomials $A\in\langle X\rangle$ which don't contain yx as a subword.

$\Rightarrow a\in \operatorname{span}_k\{x^m y^n \mid m,n\geq 0\} = k\langle X\rangle_{irr}$

- $k\langle X\rangle_{irr}$ denotes the k-submodule of irreducible elements (w.r.t. S)

- A sequence $r_1, r_2, \dots, r_n$ of reductions ($r_i = r_{A_i\sigma_i B_i}$) is final for $a\in k\langle X\rangle$ if $r_n\cdots r_2 r_1(a)\in k\langle X\rangle_{irr}$.

- $a \in k\langle X \rangle$ is reduction-finite if $\forall$ infinite sequence of reductions $r_1, r_2, \dots,$ $\exists N > 0$: r_i acts trivially on $r_{i-1} \cdots r_1(a)$ for all $i \geq N$.

$\Longrightarrow$ any max'l seq $(r_1, r_2, \dots)$ such that r_i acts nontrivially on $r_{i-1} \cdots r_1(a)$ for i, has to be finite, hence final for a.

Ex. $X = \{x, y\}$ $S = \{\sigma = (y, xy), \tau = (xyx, y)\}$

$$\begin{tikzcd}
 & xyx \arrow[dl, "r_{x\sigma x}"'] \arrow[dr, bend left, "r_{1\tau 1}"] & \\
x(xy)x \arrow[d, "r_{x^2\sigma x}"'] \arrow[drr, bend right, "r_{x\tau 1}"] & & y \arrow[d, "r_{1\sigma 1}"] \\
xx(xy)x \arrow[d] & & xy \arrow[d] \\
\vdots & & \vdots
\end{tikzcd}$$

$\Rightarrow$ xyx is not reduction-finite

- $a \in k\langle X\rangle$ is reduction-unique if a is reduction-finite & images under all final sequences of reductions are the same. This common value is denoted $r_S(a)$.

<u>Ex</u>. $X = \{x, y\}$ $\quad S = \{\sigma = (yx, x+y), \tau = (xy, x-y)\}$

(think $A = k\langle x, y \mid yx = x+y,\ xy = x-y\rangle$)

$$(x-y)x = x^2 - yx \xleftarrow{r_{1\tau x}} xyx \xrightarrow{r_{x\sigma 1}} x(x+y) = x^2 + xy$$

not reduction-unique

$$\downarrow r_{1\sigma 1} \qquad\qquad\qquad \downarrow r_{1\tau 1}$$

$$k\langle X\rangle_{irr} \ni \underline{x^2 - (x+y)} \qquad\qquad \underline{x^2 + x - y} \in k\langle X\rangle_{irr}$$

Ambiguities. Two kinds:

Overlap ambiguity: (σ, τ, A, B, C) where $\sigma, \tau \in S$, $A, B, C \in \langle X \rangle \setminus \{1\}$ such that $W_\sigma = AB$ $\quad W_\tau = BC$

$$\begin{tikzcd}
 & ABC \arrow[dl, "r_{1\sigma C}"'] \arrow[dr, "r_{A\tau 1}"] & \\
f_\sigma C \arrow[dr, dashed, "r"] & & A f_\tau \arrow[dl, dashed, "r'"'] \\
 & b &
\end{tikzcd}$$

This ambiguity is resolvable if $\exists$ compositions of reductions r, r', such that $r(f_\sigma C) = r'(A f_\tau)$

Similarly, an **inclusion ambiguity** is (σ, τ, A, B, C) where $\sigma \neq \tau \in S$, $A, B, C \in \langle X \rangle$, such that $W_\sigma = B$, $W_\tau = ABC$

$$
\begin{tikzcd}
 & ABC \arrow[dl, "r_{A\sigma C}"'] \arrow[dr, "r_{1\tau 1}"] & \\
A f_\sigma B \arrow[dr, dashed, "r"] & & f_\tau \arrow[dl, dashed, "r'"'] \\
 & b &
\end{tikzcd}
$$

Thm (Diamond Lemma, version 1)
Assume all elements are reduction-finite. TFAE:

a) All ambiguities of S are resolvable

b) $k\langle X\rangle_{irr} \hookrightarrow k\langle X\rangle \twoheadrightarrow \frac{k\langle X\rangle}{I}$

where $I = \langle W_\sigma - f_\sigma \mid \sigma \in S\rangle$,

is a k-module isomorphism, & an alg. isom. if we define

$$a \cdot b = r_S(\underbrace{ab}_{\text{in } k\langle X\rangle}) \quad \forall a, b \in k\langle X\rangle_{irr}.$$

Ex. $X = \{x, y\}$ $S = \{\sigma = (yx, xy+1)\}$

$\Rightarrow \{x^m y^n \mid m, n \geq 0\}$ is a basis for $k\langle x, y \mid yx = xy + 1\rangle \cong A_1(k)$.

Problem: Find a basis for the rational Cherednik algebra $\mathbb{H}$.