



# THE DIAMOND LEMMA.

$k$  assoc. comm. ring w  $1$

$X$  set

$\langle X \rangle$  free monoid on  $X =$  set of words in  $X$

$k\langle X \rangle$  free  $k$ -alg on  $X$ .

A **reduction system** is a subset  $S \subseteq \langle X \rangle \times k\langle X \rangle$

For  $\sigma \in S$  we write  $\sigma = (W_\sigma, f_\sigma)$

A **reduction** is a  $k$ -linear map:

$$\Gamma_{A\sigma B} : k\langle X \rangle \rightarrow k\langle X \rangle, \quad A, B \in \langle X \rangle, \sigma \in S$$
$$AW_\sigma B \mapsto Af_\sigma B$$

$$C \mapsto C, \quad C \in \langle X \rangle, C \neq AW_\sigma B$$

•  $\Gamma_{A\sigma B}$  acts trivially on  $a \in k\langle X \rangle$  if the coeff. of  $AW_\sigma B$  in  $a$  is zero.

•  $a \in k\langle X \rangle$  is irreducible (w.r.t.  $S$ ) if  $\Gamma_{A\sigma B}$  acts trivially on  $a \forall A, B \in \langle X \rangle, \sigma \in S$ .

Ex.  $X = \{x, y\}$ ,  $S = \{\sigma = (\underbrace{yx}, \underbrace{xy+1})\}$

$$\Gamma_{1\sigma 1} (yx) = xy + 1 \quad W_\sigma \quad f_\sigma$$

$$\Gamma_{x\sigma x^2} (yx^3 + 3\underbrace{xyx^3} + yx) =$$

$$xW_\sigma x^2 = yx^3 + 3\underbrace{x(xy+1)x^2} + yx$$

$\sim \Gamma_{1\sigma x^2}$

An element  $\checkmark^a$  in  $k\langle X \rangle$  is irreducible iff it's a lin. comb. of monomials  $A \in \langle X \rangle$  which don't contain  $yx$  as a subword.

$$\Rightarrow a \in \text{span}_k \{ x^m y^n \mid m, n \geq 0 \} = k\langle X \rangle_{\text{irr}}$$

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- $k\langle X \rangle_{\text{irr}}$  denotes the  $k$ -submodule of irreducible elements (w.r.t.  $S$ )
- A sequence  $r_1, r_2, \dots, r_n$  of reductions ( $r_i = r_{\alpha_i} \sigma_i B_i$ ) is **final** for  **$a \in k\langle X \rangle$**  if  $r_n \cdots r_2 r_1(a) \in k\langle X \rangle_{\text{irr}}$ .

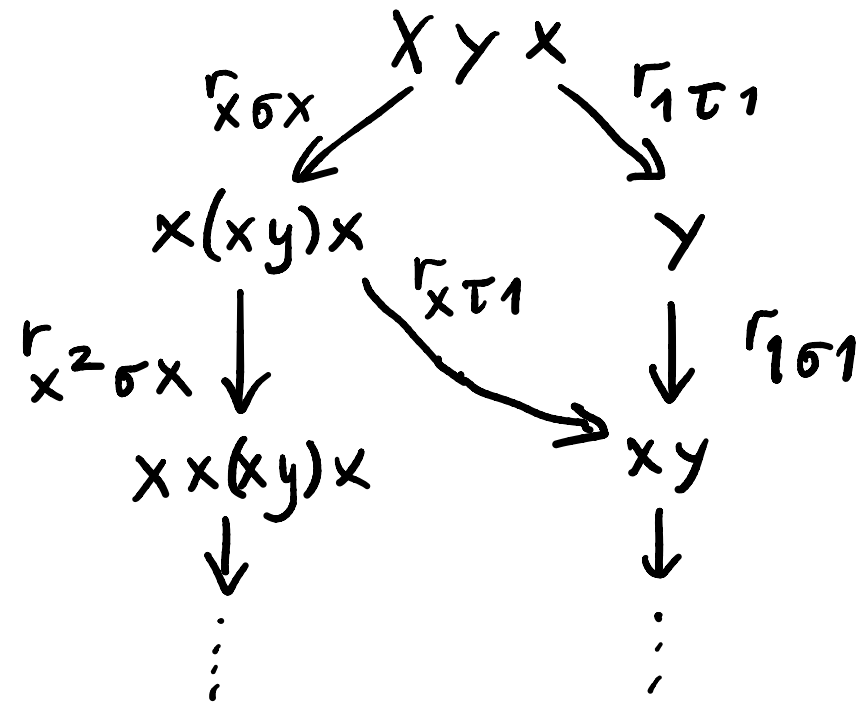
- $a \in k\langle X \rangle$  is **reduction-finite** if  
 $\nexists$  infinite sequence of reductions

$$r_1, r_2, \dots$$

$\exists N > 0$ :  $r_i$  acts trivially on  $r_{i-1} \dots r_1(a)$   
for all  $i \geq N$ .

$\Rightarrow$  any max'l seq  $(r_1, r_2, \dots)$  such that  
 $r_i$  acts nontrivially on  $r_{i-1} \dots r_1(a)$   
for  $i$ , has to be finite, hence  
final for  $a$ .

Ex.  $X = \{x, y\}$   $S = \{\sigma = (y, xy), \tau = (xyx, y)\}$



$\Rightarrow xyx$  is not reduction-finite

- $a \in k\langle X \rangle$  is **reduction-unique** if  $a$  is reduction-finite & images under all final sequences of reductions are the same. This common value is denoted  $r_s(a)$ .

Ex.  $X = \{x, y\}$   $S = \{\sigma = (yx, x+y), \tau = (xy, x-y)\}$

(Think  $A = k\langle x, y \mid yx = x+y, xy = x-y \rangle$ )

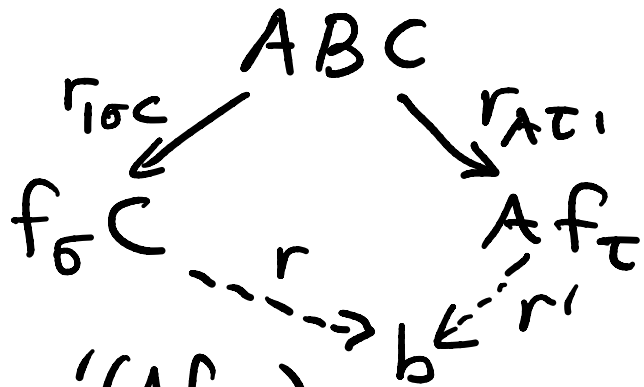
$$\begin{array}{ccc}
 & \xleftarrow{r_{\tau x}} \textcircled{xyx} \xrightarrow{r_{\sigma 1}} & \\
 (x-y)x = x^2 - yx & \text{not reduction-unique} & x(x+y) = x^2 + xy \\
 \downarrow r_{\sigma 1} & & \downarrow r_{\tau 1} \\
 k\langle x \rangle_{\text{irr}} \ni \underline{x^2 - (x+y)} & & \underline{x^2 + x - y} \in k\langle x \rangle_{\text{irr}}
 \end{array}$$

Ambiguities. Two kinds:

overlap ambiguity:  $(\sigma, \tau, A, B, C)$  where  
 $\sigma, \tau \in S$ ,  $A, B, C \in \langle X \rangle \setminus \{\epsilon\}$  such that  
 $W_\sigma = AB$      $W_\tau = BC$

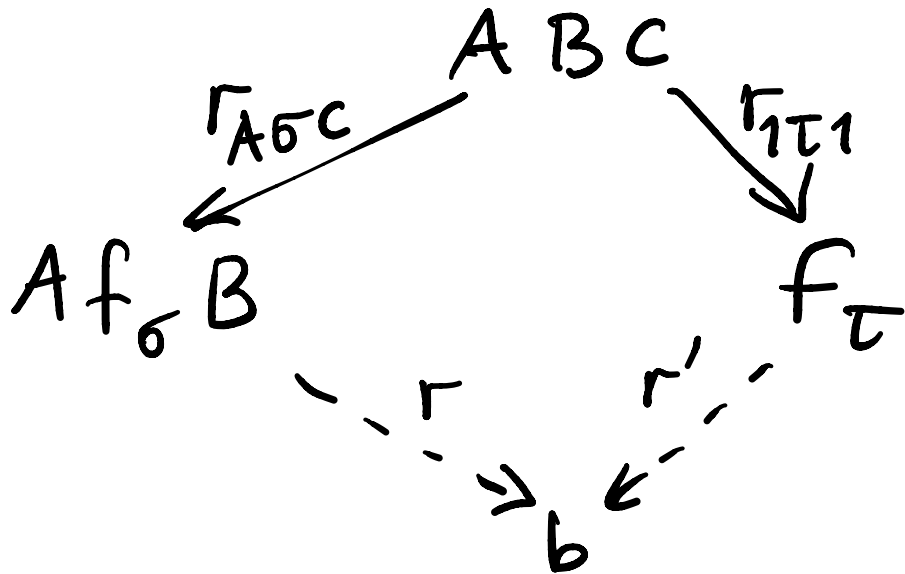
This ambiguity is resolvable  
if  $\exists$  compositions of  
reductions  $r, r'$

such that  $r(f_\sigma C) = r'(A f_\tau)$





Similarly, an inclusion ambiguity is  $(\sigma, \tau, A, B, C)$  where  $\sigma \neq \tau \in S$ ,  $A, B, C \in \langle X \rangle$ , such that  $W_\sigma = B$ ,  $W_\tau = ABC$



Thm (Diamond Lemma, version 1)

Assume all elements are reduction-finite. TFAE:

a) All ambiguities of  $S$  are resolvable

b)  $k\langle X \rangle_{irr} \hookrightarrow k\langle X \rangle \twoheadrightarrow \frac{k\langle X \rangle}{I}$

where  $I = \langle W_\sigma - f_\sigma \mid \sigma \in S \rangle$ ,

is a  $k$ -module isomorphism, & an  
alg. isom. if we define

$$a \cdot b = r_S(ab) \quad \forall a, b \in k\langle X \rangle_{irr}.$$

$\uparrow$   
in  $k\langle X \rangle$

Ex.  $X = \{x, y\}$   $S = \{\sigma = (yx, xy + 1)\}$

$\Rightarrow \{x^m y^n \mid m, n \geq 0\}$  is a basis

for  $k\langle x, y \mid yx = xy + 1 \rangle \cong A_1(k)$ .

Problem: Find a basis for the rational Cherednik algebra  $H$ .