RATIONAL CHEREDNIK ALGEBRA.

(= DOUBLY DEGENERATE $\underbrace{\text{DOUBLE AFFINE HECKE ALGEBRA}}_{\text{DAHA}}$)

$$\mathbb{H} = \mathbb{C}\langle x, t, s\rangle / (\text{Rels}), \quad \text{Rels}: \begin{cases} tx - xt = 1 + s \\ sx + xs = 0 \\ st + ts = 0 \\ s^2 = 1 \end{cases}$$

CLAIM: $\exists \rho : \mathbb{H} \longrightarrow \mathrm{End}(\mathbb{C}[x])$

$$x \longmapsto \text{(mult. by) } x$$

$$s \longmapsto (p(x) \mapsto p(-x))$$

$$t \longmapsto \left(p(x) \mapsto \frac{dp}{dx} + \frac{p(x) - p(-x)}{2x}\right)$$

PROOF (SKETCH). DEFINE

$$\tilde{\rho} : \{x, s, t\} \to \mathrm{End}(\mathbb{C}[x]) \quad \text{(USING RHS OF } \rho\text{)}$$

$$\Rightarrow \rho' : \mathbb{C}\langle x, s, t\rangle \to \mathrm{End}(\mathbb{C}[x])$$

CHECK RELS:

$$\rho'(tx - xt - (1+s))(p(x)) =$$

$$= t(xp(x)) - x \cdot t(p(x)) - (p(x) + p(-x))$$

$$= \frac{d}{dx}(xp(x)) + \frac{xp(x) + xp(-x)}{2x} - x\left(\frac{dp}{dx} + \frac{p(x) - p(-x)}{2x}\right)$$

$$- (p(x) + p(-x)) =$$

$$= p(x) + \cancel{x p'(x)} + \frac{1}{2}\left(\cancel{p(x)} + p(-x)\right) - \cancel{x p'(x)} - \frac{1}{2}\left(\cancel{p(x)} - p(-x)\right)$$

$$- \left(p(x) - p(-x)\right)$$

$$= p(x) + p(-x) - \left(p(x) - p(-x)\right) = 0$$

Remark $\mathbb{H}$ IS ACTUALLY $\mathbb{H}(S_2)$

SYMM. GRP.

$$\mathbb{H}(S_n) = \frac{\mathbb{C}\langle x_1, \ldots, x_{n-1}, t_1, \ldots, t_{n-1}, s_1, \ldots, s_{n-1}\rangle}{\langle \text{RELS} \rangle}$$

SURVEY PAPER: ROUQUIER

GRADATION

RECALL:

$$\begin{cases} tx - xt - (1+s) = 0 & \text{degree } 0 \\ sx + xs = 0 & \text{degree } 1 \\ st + ts = 0 & \text{degree } -1 \\ s^2 - 1 = 0 & \text{degree } 0 \end{cases}$$

$X = \{t, x, s\}$ $\quad d(t) = -1,\ d(x) = 1,\ d(s) = 0$

$\Longrightarrow$ $\mathbb{H}$ IS A $\mathbb{Z}$-GRADED ALGEBRA.

IN FACT: $\mathbb{H}$ IS INTERNALLY GRADED, THAT IS, $\exists\, d \in \mathbb{H}$ SUCH THAT

$$\mathbb{H}_n = \{u \in \mathbb{H} \mid \overset{ad_d(u)}{\overset{\|}{[d, u]}} = n \cdot u\} \quad \forall n \in \mathbb{Z}.$$

(cf.: $[x\partial, x^n] = n x^n$ IN $A_1(\mathbb{C})$)

ASIDE

A ANY ALG, $x \in A$.

THEN $\quad ad_x : A \to A$

$$a \mapsto [x,a] = xa - ax$$

IS A DERIVATION:

$$[x, ab] = [x,a]b + a[x,b]$$

SUCH DERIVATIONS ARE CALLED INNER.

$\mathbb{Z}$-GRADED ALGEBRAS HAVE A DEGREE DERIVATION: $\quad D|_{A_n} = n \cdot \mathrm{Id}|_{A_n}$.

$\mathbb{Z}$-GRADED ALG IS INTERNALLY GRADED $\Leftrightarrow$ D IS INNER

$A_1(\mathbb{C})$: $\quad D = \mathrm{ad}_{x\partial}$

$\mathbb{H}$: Try $D = \mathrm{ad}_d$ WITH $d = xt$

$[xt, x] = x[t, x] = x\cdot(1+s) \neq 1\cdot x$

$[xt, t] = [x, t]t = -(1+s)t \neq -1\cdot t$

$[xt, s] = xts - sxt$

$= x\cdot s\cdot(-t) - sxt = s\cdot(-x)(-t) - sxt$

DOESN'T WORK!

$= 0 = 0\cdot s$

TRY $d = \frac{1}{2}(xt + tx)$: $\quad \frac{1}{2}[xt + tx, x] =$

$= \frac{1}{2}(x\cdot(1+s) + (1+s)x) = 1\cdot x$

$$
\begin{aligned}
&[xt, x] = x[t, x] = x\cdot(1+s) \\
+\ &[tx, x] = [t, x]x = (1+s)x \\
\hline
&[xt+tx, x] = x + xs + x + sx = 2x
\end{aligned}
$$

$= 0$

DIVIDE BY 2.

FOURIER

BONUS FACT $\exists$ AUTOMORPHISM $\theta : \mathbb{H} \to \mathbb{H}$

$$\theta(x) = t, \quad \theta(t) = -x, \quad \theta(s) = s$$

$$\Rightarrow \left[\tfrac{1}{2}(xt+tx), t\right] = \left[-\theta\left(\tfrac{1}{2}(xt+tx)\right), \theta(x)\right] = \underbrace{\theta(-x)}_{=-t}$$

READING ASSIGNMENT:

G. BERGMAN "DIAMOND LEMMA FOR RING THEORY"

ADV. MATH. 1978

SECTION 1, (2.1), 3.

PBW.

$\Rightarrow$ BASIS FOR $U(\mathfrak{g})$