FILTERED ALGEBRAS.

A FILTRATION F ON AN ALG A IS A FAMILY OF SUBSPACES $F = (F_n A)_{n \in \mathbb{N}}$ SUCH THAT, PUTTING $A_{(n)} = F_n A$,

1) $A_{(0)} \subseteq A_{(1)} \subseteq A_{(2)} \subseteq \cdots$
2) $A = \bigcup_{n \geq 0} A_{(n)}$
3) $A_{(m)} A_{(n)} \subseteq A_{(m+n)}$
4) $1_A \in A_{(0)}$

<u>Ex</u>. $A_{(n)} = A \ \forall n$ (TRIVIAL FILTRATION).

<u>Ex</u>. $\mathbb{C}[x]_{(n)} = \mathbb{C}1 + \mathbb{C}x + \cdots + \mathbb{C}x^n$

EX. A ANY ALG,

$X = \{a_i\}_{i \in I}$ ANY GENERATING SET

$d : I \to \mathbb{N}$ ANY "DEGREE FUNCTION"

(OR $d : X \to \mathbb{N}$) $\deg a_i = d(i)$

DEFINE $A_{(n)} = \mathrm{Span}\left\{ a_{i_1} a_{i_2} \cdots a_{i_k} \;\middle|\; \begin{array}{l} i_j \in I \\ \sum_{1 \le j \le k} d(i_j) \le n \end{array} \right\}$

THIS DEFINES A FILTRATION ON A.

Ex. $A = A_1(\mathbb{C}) = \mathbb{C}\langle \partial, x\rangle / \langle \partial x - x\partial - 1\rangle$

$X = \{x, \partial\}$ $\quad d(x) = 1 \quad d(\partial) = 1$

$A_{(0)} = \mathbb{C}1$, $\quad A_{(1)} = \mathbb{C}x + \mathbb{C}\partial + \mathbb{C}1$

$$A_{(2)} = \mathbb{C}x\partial + \mathbb{C}x^2 + \mathbb{C}\partial^2 + \mathbb{C}\underbrace{\partial x}_{= x\partial + 1} + A_{(1)}$$

$$= \mathbb{C}x^2 + \mathbb{C}x\partial + \mathbb{C}\partial^2 + A_{(1)}$$

$$A_{(n)} = \bigoplus_{k+l \leq n} \mathbb{C}x^k\partial^l$$

GRADED ASIDE

$$V = \bigoplus_{n\in\mathbb{N}} V_n$$

$v \in V$ IS HOMOGENEOUS IF $v \in \bigcup_{n\in\mathbb{N}} V_n$

$$A = \bigoplus_{n\in\mathbb{N}} A_n$$

$$I = \langle a_{n_1}, a_{n_2}, \dots, a_{n_k} \rangle = Aa_{n_1}A + Aa_{n_2}A + \dots + Aa_{n_k}A, \quad a_{n_i} \in A_{n_i} \ \forall i$$

<u>THEN</u> $I = \bigoplus_{n\in\mathbb{N}} (A_n \cap I)$

<u>EX</u> $A = \mathbb{k}\langle x, \partial \rangle$ FREE ALG. DECLARING $\deg x = 1 = \deg \partial$ DEFINES AN $\mathbb{N}$-GRADATION ON A. $x\partial - \partial x \in A_2$ $\quad \partial x - x\partial - 1 \notin \bigcup_{n\geq 0} A_n$

$a_n \in A_n \cap I$, $A = \bigoplus_{n \geq 0} A_n$

$$A a_n A = \operatorname{Span}\{ b a_n c \mid b, c \in A \}$$

$$= \left\{ \sum_{k,l \geq 0} \underbrace{b_k a_n c_l}_{\in A_{k+n+l} \cap I} \;\middle|\; \begin{array}{l} b_k \in A_k \\ c_l \in A_l \end{array} \right\}$$

$$(A a_n A) \cap A_m = \sum_{k+n+l=m} A_k a_n A_l$$

$$\Rightarrow I = \bigoplus_{n \geq 0} (A_n \cap I)$$

END ASIDE

Ex. $X = \{e, f, h\} \subset U(\mathfrak{sl}_2)$

$d(e) = d(f) = d(h) = 1 \Rightarrow$ FILTRATION.

PROBLEM SHOW THAT

$$\mathrm{Span}\left\{ h^k f^l e^m \;\middle|\; \begin{array}{l} k, l, m \geq 0 \\ k + l + m \leq N \end{array} \right\} = U(\mathfrak{sl}_2)_{(N)}$$

(Ex. $ef = fe + \underbrace{h} \in \mathrm{LHS}$)

ASSOCIATED GRADED ALG.

$(A, F=(A_{(n)})_{n\in\mathbb{N}})$ FILTERED ALG.

DEFINE THE VECTOR SPACE

$$\operatorname{gr} A = \operatorname{gr}_F A = \bigoplus_{n=0}^{\infty} \underbrace{A_{(n)}/A_{(n-1)}}_{\text{v. sp.}} , \quad A_{(-1)}=0$$

NONZERO ELEMENTS OF $A_{(n)}/A_{(n-1)}$ ARE CALLED

LEADING TERMS OF DEGREE n.

DEFINE PRODUCT ON $\operatorname{gr} A$ BY:

$$(a_n + A_{(n-1)})(b_m + A_{(m-1)}) = \overbrace{a_n b_m + A_{(n+m-1)}}^{\in A_{(n+m)}/A_{(n+m-1)}}$$

$\mathrm{gr}\, A$ IS A GRADED ALG, CALLED THE ASSOCIATED GRADED ALG TO (A,F).

EX. $A = A_1(\mathbb{C})$ $\quad d(x) = d(\partial) = 1$

$$A_{(1)}/A_{(0)} = \mathbb{C}\bar{x} \oplus \mathbb{C}\bar{\partial}$$

$$A_{(n)}/A_{(n-1)} = \bigoplus_{k+l=n} \mathbb{C}\,\bar{x}^k \bar{\partial}^l$$

$$\bar{\partial}\bar{x} - \bar{x}\bar{\partial} = \partial x - x\partial + A_{(1)}$$
$$= 1 + A_{(1)}$$
$$= 0 + A_{(1)} \qquad 1 \in A_{(0)} \subset A_{(1)}$$

SO $\mathrm{gr}\, A_1(\mathbb{C}) \cong \mathbb{C}[\bar{x}, \bar{\partial}]$ POL. ALG.

APPLICATIONS.

① TH. 1.6.9: $\left.\begin{array}{l} A \text{ FILTERED ALG} \\ \mathrm{gr}\, A \text{ LEFT NOETHERIAN} \\ \quad \text{(OR RIGHT)} \end{array}\right\} \Longrightarrow A$ IS LEFT NOETH. (OR RIGHT)

COR. $A_1(\mathbb{C})$ IS LEFT & RIGHT NOETH.

(OR ENTIRE)

② AN ALG A IS A DOMAIN IF $ab=0 \Longrightarrow a=0$ OR $b=0$.

TH. $\left.\begin{array}{l} A \text{ FILTERED ALG} \\ \mathrm{gr}\, A \text{ DOMAIN} \end{array}\right\} \Longrightarrow A$ IS A DOMAIN.

③ PART II OF McC.-R : DIMENSIONS.

KRULL DIMENSION $\mathrm{Kr.dim}(A)$

GELFAND-KIRILLOV DIM. $\mathrm{GKdim}(A)$

GLOBAL DIMENSION $\mathrm{gldim}(A)$

A = FILTERED:

FACTS: i) $\mathrm{Kr.dim}(A) \leq \mathrm{Kr.dim}(\mathrm{gr} A)$

ii) $\mathrm{GKdim}(A) \geq \mathrm{GKdim}(\mathrm{gr} A)$

Th 7.6.18 iii) $\mathrm{gl\,dim}(A) \leq \mathrm{gl.dim}(\mathrm{gr} A)$

COR. $\mathrm{gl\,dim}(A_1(\mathbb{C})) \leq \mathrm{gl.dim}(\mathbb{C}[\bar{x}, \bar{\partial}]) = 2$

(IN FACT = HOLDS)

PROBLEM. $U(\mathfrak{sl}_2)$ $\quad d(e)=d(f)=d(h)=1$

SHOW THAT

1) $\mathrm{gr}\, U(\mathfrak{sl}_2)$ IS GENERATED BY $\bar{e}, \bar{f}, \bar{h} \in U(\mathfrak{sl}_2)_{(1)}/U(\mathfrak{sl}_2)_{(0)}$

2) $\mathrm{gr}\, U(\mathfrak{sl}_2)$ IS COMMUTATIVE.

CONCLUDE THAT $U(\mathfrak{sl}_2)$ IS LEFT & RIGHT NOETHERIAN.

(PBW THM: $\mathrm{gr}\, U(\mathfrak{sl}_2) \cong \underbrace{\mathbb{C}[\bar{e}, \bar{f}, \bar{h}]}_{\text{POL. ALG.}}$)