DEGREE DERIVATIONS.

ASSUME char $\mathbb{k} = 0$ SO THAT $\mathbb{Z} \hookrightarrow \mathbb{k}$

LET $\{A_n\}_{n\in\mathbb{Z}}$ BE A $\mathbb{Z}$-GRADATION ON AN ALGEBRA A. DEFINE A LINEAR MAP

$$d : A \longrightarrow A$$

by

$$d|_{A_n} = n \cdot Id_{A_n}$$

$a_i \in A_i$

Ex $d(a_{-2} + a_3 + a_5) = -2a_{-2} + 3a_3 + 5a_5$

CLAIM: d IS A DERIVATION ON A

PF. WTS $d(ab) = d(a)b + a\,d(b)$ $\forall a, b \in A$.

BY LINEARITY, SUFFICES TO SHOW FOR $a, b \in \bigcup_{n\in\mathbb{Z}} A_n$.

SAY $a \in A_n$ $b \in A_m$. THEN $ab \in A_{n+m}$

SO

$$d(ab) = (n+m)ab$$

WHILE

$$d(a)b + a\,d(b) = nab + mab$$

Q.E.D.

<u>DEF</u> d IS CALLED THE DEGREE DERIVATION ON A W.R.T. $\{A_n\}_{n \in \mathbb{Z}}$.

<u>DEF</u> AN M-GRADED ALGEBRA IS AN ALG A TOGETHER WITH AN M-GRADATION $\{A_m\}_{m \in M}$ ON A.

CONVERSELY, SUPPOSE

1) A IS AN ALG
2) $d: A \to A$ IS A DERIVATION SUCH THAT d IS DIAGONALIZABLE WITH INTEGER EIGENVALUES.

DEFINE $A_n = E_n(d) = \{a \in A \mid d(a) = na\}, n \in \mathbb{Z}$

<u>CLAIM</u> $\{A_n\}_{n\in\mathbb{Z}}$ IS A $\mathbb{Z}$-GRADATION ON A.

<u>PF</u>. BY 2), $A = \bigoplus_{n\in\mathbb{Z}} A_n$

BY LEIBNIZ RULE FOR d: $A_m A_n \subseteq A_{m+n}$.

LASTLY $d(1) = 0 \implies 1_A \in A_0$. <u>QED</u>.

EX. WEYL ALG $A_1(\Bbbk)$ HAS A $\mathbb{Z}$-GRADATION DETERMINED BY $\deg(x)=1$ $\deg(\partial)=-1$:

$$A_1(\Bbbk) = \Bbbk\langle x,\partial\rangle / \langle \underbrace{\partial x - x\partial - 1}_{\in \Bbbk\langle x,\partial\rangle_0} \rangle$$

DEFINE

$d: \{x,\partial\} \to \Bbbk\langle x,\partial\rangle$ BY

$d(x) = 1\cdot x \qquad d(\partial) = -\partial$

d EXTENDS TO A DERIVATION $d: \Bbbk\langle x,\partial\rangle \to \Bbbk\langle x,\partial\rangle$

WTS $d(I) = 0$, $I = \langle \partial x - x\partial - 1\rangle$

BY LEIBNIZ RULE, SUFFICES TO CHECK

$d(\partial x - x\partial - 1) \in I$. HAVE:

$d(\partial x - x\partial - 1) = \cdots = 0 \implies \underbrace{d: A_1(\Bbbk) \to A_1(\Bbbk)}_{\text{WELL-DEFINED}}$.

GRADED IDEALS & QUOTIENTS

A ALG, M MONOID, $\{A_m\}_{m\in M}$ AN M-GRADATION ON A.

DEF A (LEFT OR RIGHT OR 2-SIDED) IDEAL $I \subset A$ IS GRADED IF

$$I = \bigoplus_{m\in M} (A_m \cap I)$$

NOTE: SUM IS ALWAYS DIRECT & $\supseteq$ HOLDS.

EX. $A = \mathbb{k}[x] = \bigoplus_{n\in\mathbb{N}} \mathbb{k}\cdot x^n$ $\mathbb{N}$-GRADATION

$I = (x-1) = \{\, p(x)\cdot(x-1) \mid p(x)\in A \,\} \Rightarrow A_m \cap I = 0 \quad \forall m \in \mathbb{N}$

PROBLEM: A ALG, M MONOID, $\{A_m\}_{m\in M}$ M-GRADATION & $I \subseteq A$ AN IDEAL.

SHOW THAT TFAE:

1) I IS GRADED.

2) WHENEVER

$$a_{m_1} + a_{m_2} + \dots + a_{m_k} \in I$$

FOR SOME $a_{m_i} \in A_{m_i}$ $m_i \neq m_j \ \forall i \neq j$

THEN $a_{m_i} \in I \ \forall i = 1, \dots, k$.

NOTATION AS ABOVE. A, M, I

LEMMA IF I IS GRADED THEN A/I HAS A NATURAL M-GRADATION:

$$A/I = \bigoplus_{m \in M} (A/I)_m \qquad \neq \{a+i \mid a \in A_m, i \in I\}$$

$$(A/I)_m = \{a + I \mid a \in A_m\} = \text{"}A_m + I\text{"} \simeq \overline{A_m}$$

PF STRAIGHTFORWARD.

<u>REMARK</u>. IF $I \subseteq A$ IS GRADED,

PUT $I_m = A_m \cap I$ SO THAT $I = \bigoplus_{m \in M} I_m$

<u>CLAIM</u>: $\boxed{\{a+I \mid a \in A_m\} \cong A_m/I_m \text{ as vec. space}}$

$$a + I \longmapsto a + I_m$$

$$\left.\begin{array}{l} a+I = b+I \Rightarrow a-b \in I \\ a-b \in A_m \end{array}\right\} \Rightarrow a-b \in I_m$$

<u>COR</u>. $A/I \cong \bigoplus_{m \in M} (A_m/I_m)$

I 2-SIDED: $(a_m + I_m)\cdot(b_n + I_n) = a_m b_n + I_{mn}$

GWAs.

[REMARK: $YX = t \iff XY = \sigma(t)$

$\Longrightarrow$: $X(YX) = (XY)X$ BY ASSOCIATIVITY

$\quad\; \|$

$X \cdot t = \sigma(t) X$

$\Longrightarrow (XY)X = \sigma(t) X$

$\Longrightarrow (XY - \sigma(t))X = 0 \Longrightarrow XY = \sigma(t)$

SO: IF A IS AN ALGEBRA CONTAINING A SUBALG R, AND $X, Y \in A$, $t \in R$ SUCH THAT $YX = t$, $X r = \sigma(r) X$ $\forall r \in R$, $\cancel{rX = 0 \Longrightarrow r = 0}$. THEN $XY = \sigma(t)$.

PROBLEM. LET $A = R(\sigma, t)$ BE A GWA

1) DESCRIBE THE CENTRALIZER OF R IN A:

$$C_A(R) = \{ a \in A \mid a \cdot r = r \cdot a \ \forall r \in R \}$$

2) DESCRIBE $Z(A) = \{ a \in A \mid ab = ba \ \forall b \in A \}$

a) ASSUMING σ HAS INFINITE ORDER.

b*) WITHOUT ANY ASSUMPTION ON σ.