LHS

$$(r_i)_i \, a x = (r_i \cdot a)_i \, x$$

$$= \left(\sigma(r_{i-1} \cdot a) + \delta(r_i \cdot a) \right)_i$$

RHS

$$(r_i)_i \left(x \sigma(a) + \delta(a) \right) =$$

$$= (r_i)_i \, x \, \sigma(a) + (r_i)_i \, \delta(a)$$

$$= \left(\sigma(r_{i-1}) + \delta(r_i) \right)_i \cdot \sigma(a) +$$

$$+ \left(r_i \, \delta(a) \right)_i =$$

$$= \left(\sigma(r_{i-1}) \sigma(a) + \delta(r_i) \sigma(a) + \right.$$

$$\left. + \, r_i \, \delta(a) \right)_i$$

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MORE ON GWAs.

R INTEGRAL DOMAIN $/\mathbb{k}$ = COMM. (ASSOC. w/1) $\mathbb{k}$-ALG WITH $rs=0 \Rightarrow r=0 \vee s=0$

$t \in R$, $t \neq 0$

$\sigma \in \mathrm{Aut}_{\mathbb{k}\text{-Alg}}(R)$

GWA: $$A = R(\sigma,t) = \frac{R\langle X,Y\rangle}{\left\langle \begin{array}{c} YX-t,\ XY-\sigma(t) \\ Xr-\sigma(r)X,\ Yr-\sigma^{-1}(r)Y \\ r\in R \end{array}\right\rangle}$$

EX: $\mathbb{k}[u]\left(\sigma(u)=u-1,\ t=u\right) \cong A_1(\mathbb{k})$ Weyl Algebra.

Ex. $R = \mathbb{C}[h, C]$ POL. ALG IN 2 VARS.

$\sigma : R \to R$, $\sigma(h) = h-2$, $\sigma(C) = C$

$t = C + (\alpha h^2 + \beta h + \gamma)$

FOR CERTAIN $\alpha, \beta, \gamma \in \mathbb{C}$

$$R(\sigma, t) \cong U(\mathfrak{sl}_2(\mathbb{C}))$$

$$h \longleftarrow h$$

$$X \longleftarrow e$$

$$Y \longleftarrow f$$

$C \longmapsto C$ = Casimir element $= ef + fe + \frac{1}{2}h^2$

Ex. $t = 1$

$$\left.\begin{array}{l} XY = \sigma(1) = 1, \quad YX = 1 \end{array}\right\} Y = X^{-1}$$

$$R(\sigma, t=1) \cong R[X, X^{-1}; \sigma] =$$

$$= \left\{ \sum_{n \in \mathbb{Z}} r_n X^n \,\middle|\, \begin{array}{l} r_n \in R \\ a.a.\ r_n = 0 \end{array} \right\}$$

$$Xr = \sigma(r) X \implies X^n \cdot r = \sigma^n(r) X^n \quad \forall n \in \mathbb{Z}$$

SKEW LAURENT POL. ALGEBRAS ARE GWAs.

CONNECTION TO ITERATED SKEW POL RINGS:

THM: $R(\sigma, t) \cong R[Y; \sigma^{-1}][X; \hat{\sigma}, \delta] \big/ \langle YX - t \rangle$

$\hat{\sigma} \in \mathrm{Aut}_{\Bbbk}(R[Y; \sigma^{-1}])$

$$\begin{cases} \hat{\sigma}(r) = \sigma(r) & \forall r \in R \\ \hat{\sigma}(Y) = Y \end{cases}$$

$$\begin{cases} \delta(r) = 0 & \forall r \in R \\ \delta(Y) = \sigma(t) - t \end{cases}$$

$Xr = \sigma(r) X$, $\quad XY = \underbrace{Y}_{\hat{\sigma}(Y)} X + \underbrace{\sigma(t) - t}_{\delta(Y)}$

COR. IF R IS (LEFT & RIGHT) NOETHERIAN THEN SO IS ANY GWA $R(\sigma, t)$.

C.F. HILBERT BASIS THM:

R COMM NOETH $\Rightarrow$ $R[x]$ IS NOETH.

BOOK: R NOETH $\Rightarrow$ $R[x; \sigma, \delta]$ NOETH.

GRADED ALGEBRAS.

LET A BE AN ALGEBRA & M BE A MONOID $(M, \cdot, 1)$.

AN M-GRADATION ON A IS A FAMILY OF VECTOR SUBSPACES $(A_m)_{m \in M}$ SUCH THAT

1) $A = \bigoplus_{m \in M} A_m$

2) $A_m \cdot A_n \subset A_{m \cdot n}$ $\qquad \operatorname{span}\{a \cdot b \mid a \in A_m, b \in A_n\}$

3) $1_A \in A_{1_M}$

EX. $A = \mathbb{k}[x]$ POL RING.

$M = (\mathbb{N}, +, 0)$

THEN $(A_n = \mathbb{k}x^n)_{n \in \mathbb{N}}$ IS AN $\mathbb{N}$-GRADATION OF A: $x^m \cdot x^n = x^{m+n}$

EX. $A = R(\sigma, t)$ GWA.

$M = (\mathbb{Z}, +, 0)$

$$A_n = \begin{cases} R \cdot x^n, & n > 0 \\ R \cdot 1, & n = 0 \\ R \cdot y^{|n|}, & n < 0 \end{cases}$$

$\boxed{\text{PROBLEM: PROVE THAT } (A_n)_{n \in \mathbb{Z}} \text{ IS A } \mathbb{Z}\text{-GRADATION ON } A.}$