# SKEW POLYNOMIAL RINGS.

[Ref: McConnell Robson Chapter 1 §§2–4]

$A$ ALGEBRA/$\Bbbk$.

$\sigma, \delta : A \to A$ FUNCTIONS.

$$A[x;\sigma,\delta] \stackrel{\text{DEF}}{=} \frac{A\langle x\rangle}{\langle x\cdot a - (\sigma(a)x + \delta(a)) \mid a \in A\rangle}$$

$\underline{\text{Ex}}$. $\sigma = \mathrm{Id}_A$, $\delta = 0$

$$\implies A[x;\mathrm{Id}_A, 0] \cong A[x]$$

CONDITION: $A[x;\sigma,\delta]$ IS FREE AS A LEFT A-MODULE ON THE SET $\{x^n\}_{n=0}^{\infty}$:

$$A[x;\sigma,\delta] = \bigoplus_{n=0}^{\infty} Ax^n$$

EX SUPPOSE $\sigma(1_A) = 0$, $\delta(1_A) = 0$

THEN $x = x \cdot 1_A = \sigma(1_A)x + \delta(1_A)$
$= 0x + 0 = 0$

CONDITION FAILS HERE.

SUPPOSE CONDITION HOLDS THEN IN $A[x;\sigma,\delta]$ WE HAVE

$$x(a_1+a_2) = \sigma(a_1+a_2)x + \delta(a_1+a_2)$$

$$\|$$

$$xa_1 + xa_2 = (\sigma(a_1)+\sigma(a_2))x + \delta(a_1)+\delta(a_2).$$

$$\Rightarrow \left(\sigma(a_1+a_2) - (\sigma(a_1)+\sigma(a_2))\right)x + \left(\delta(a_1+a_2) - (\delta(a_1)+\delta(a_2))\right)1 = 0$$

SINCE $\{1,x\} \subset \{x^n\}_{n=0}^{\infty}$ IS LIN. INDEP OVER A ON THE LEFT, $\sigma, \delta$ ARE <u>ADDITIVE</u>

SIMILARLY $\forall \lambda \in \mathbb{K}, a \in A$

$$x(\lambda a) = \cdots$$

$$\parallel$$

$$\lambda \cdot (xa) = \cdots$$

$$\Rightarrow \sigma(\lambda a) = \lambda \sigma(a)$$

$$\delta(\lambda a) = \lambda \delta(a)$$

So $\sigma$ & $\delta$ ARE $\mathbb{K}$-LINEAR MAPS.

AND $\forall a_1, a_2 \in A$:

$$x(a_1 a_2) = \underline{\underline{\sigma(a_1 a_2)}}\, x + \underline{\underline{\delta(a_1 a_2)}}$$

$$\|$$

$$(x a_1) a_2 = \big(\sigma(a_1) x + \delta(a_1)\big) a_2 =$$

$$= \sigma(a_1)(x a_2) + \delta(a_1) a_2$$

$$= \sigma(a_1) \cdot \big( \sigma(a_2) x + \delta(a_2) \big) + \delta(a_1) a_2$$

$$= \underline{\underline{\sigma(a_1)\sigma(a_2)}}\, x + \big( \underline{\underline{\sigma(a_1)\delta(a_2) + \delta(a_1) a_2}} \big)$$

$\sigma(1_A) = 1_A \qquad \delta(1_A) = 0$

$$x = x 1_A = \sigma(1_A) x + \delta(1_A)$$
$$\| $$
$$1_A \cdot x + 0$$

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DEF IF $\sigma : A \to A$ IS AN ALGEBRA MAP, A $\sigma$-DERIVATION $\delta : A \to A$ IS A LINEAR MAP SATISFYING

$$\delta(a_1 a_2) = \sigma(a_1)\delta(a_2) + \delta(a_1) a_2$$

$\forall a_1, a_2 \in A$. $\quad (\Rightarrow \delta(1_A) = 0)$

THM. $A[x;\sigma,\delta]$ IS FREE AS A LEFT A-MODULE ON $\{x^n\}_{n=0}^{\infty}$ IFF

1) $\sigma : A \to A$ IS ALGEBRA ENDOMORPHISM.

2) $\delta : A \to A$ IS A $\sigma$-DERIVATION.

PF ($\Longrightarrow$) DONE!

($\Longleftarrow$): LESS OBVIOUS! (READ BOOK.)

"QED"

DEF $A[x;\sigma,\delta]$ SKEW POLYNOMIAL RING WHEN THESE CONDITIONS HOLD.

Ex $\delta = 0$

$$A[x;\sigma] = A[x;\sigma,0]$$

$$= \left\{ \sum_{n=0}^{\infty} a_n x^n \;\middle|\; a_n \in A \text{ (ALMOST ALL} = 0) \right\}$$

$$xa = \sigma(a)x \quad \forall a \in A$$

SUB-EX: $A = \mathbb{k}[t]$ $\quad \sigma'(t) = t^2+1$

$\sigma' : \{t\} \to A$

& EXTEND TO ALG MAP $\sigma$

$$\sigma(t^3) = \sigma(t)^3 = (t^2+1)^3$$

IN $A[x;\sigma]$: $\boxed{xt = (t^2+1)x}$

**Ex** $A = \mathbb{k}[t]$, $\sigma = \mathrm{Id}_A$, $\delta(t) = 1$

$$(\Rightarrow \delta(t^2) = t\,\delta(t) + \delta(t)\,t$$
$$= t \cdot 1 + 1 \cdot t = 2t)$$

**PROBLEM**: SHOW THAT ANY FUNCTION

$$\delta' : \{x_1, \dots, x_n\} \longrightarrow F_{\mathbb{k}}(\{x_1, \dots, x_n\})$$

EXTENDS TO A (UNIQUE) DERIVATION

$$\delta : F_{\mathbb{k}}(\{x_1, \dots, x_n\}) \to F_{\mathbb{k}}(\{x_1, \dots, x_n\})$$

$$x \cdot t = t \cdot x + 1 \Rightarrow \mathbb{k}[t]\left[x; \mathrm{Id}, \frac{d}{dt}\right]$$

$\cong$ Weyl Alg.

REMARK

$$\delta(t) = p(t)$$

$$\delta = p(t)\frac{d}{dt} : q(t) \mapsto p(t)q'(t)$$

1) ON $t$ THEY COINCIDE

2) UNIQUENESS

SHOW THE FORGETFUL FUNCTOR

$\Theta : A\langle X\rangle\text{-Mod} \longrightarrow A\text{-Mod}$

HAS A LEFT ADJOINT & DESCRIBE IT.

$$\mathrm{Hom}_{A\langle X\rangle\text{-Mod}}\big(\,?(M), N\big) \cong \mathrm{Hom}_A(M, \Theta N)$$

$M \in A\text{-Mod}$

$N \in A\langle X\rangle\text{-Mod}$