ADJOINING GENERATORS.

LET A BE AN ALGEBRA, AND X BE A SYMBOL (= INDETERMINATE = LETTER). LET F BE FREE ALGEBRA ON THE SET $A \cup \{X\}$. LET R BE THE SET OF RELATIONS:

$1 - 1_A = 0$ NEEDS TO BE INCLUDED TOO.

$1_A \cdot a - a = 0$

$$\begin{cases} a+b-(a+b)_A = 0 & \forall a,b \in A \\ a \cdot b - (a \cdot b)_A = 0 & \forall a,b \in A \\ \lambda \cdot a - (\lambda \cdot a)_A = 0 & \forall a \in A \ \forall \lambda \in \mathbb{k} \end{cases}$$

$(\cdot)_A$ MEANS COMPUTE IN A.

DEF $\mathbb{k}\langle A \cup \{X\} \mid R = 0\rangle$ IS THE ALGEBRA OBTAINED FROM A BY ADJOINING X. NOTATION: $A\langle X\rangle$

EX 1) $A = \mathbb{k} \implies \mathbb{k}\langle X\rangle = F_{\mathbb{k}}(\{X\}) = \mathbb{k}[X]$

2) $A = F_{\mathbb{k}}(\{Y_1, \ldots, Y_n\})$

$$A\langle X\rangle = F_{\mathbb{k}}(\{Y_1, \ldots, Y_n, X\})$$

TYPICAL ELEMENT OF $A\langle X\rangle$ IS A LIN. COMB. OF PRODUCTS

$$a_1 X a_2 X \cdots X a_n$$

ASIDE

$$A \subset B \qquad \text{TWO ALGEBRAS WITH UNIT}$$
$$\begin{matrix} \psi & & \psi \\ 1_A & & 1_B \end{matrix}$$

EX

$$\begin{bmatrix} \mathbb{k} & 0 \\ 0 & 0 \end{bmatrix} \subset \begin{bmatrix} \mathbb{k} & \mathbb{k} \\ \mathbb{k} & \mathbb{k} \end{bmatrix} = M_2(\mathbb{k})$$

$$1_A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad 1_B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

A IS A UNITAL SUBALGEBRA OF B

IF $1_A = 1_B$.

IF WE WANT A TO BE A UNITAL SUBALGEBRA OF $A\langle X\rangle$ WE MUST INCLUDE THE RELATION

$$1 - 1_A$$

EMPTY WORD = UNIT IN FREE ALG

UNIT IN A

END ASIDE

UNIVERSAL PROPERTY

GIVEN AN ALGEBRA MAP

$$\varphi : A \longrightarrow B$$

AND ANY ELEMENT $b \in B$, THERE IS A UNIQUE ALGEBRA MAP

$$\tilde{\varphi} : A\langle X\rangle \longrightarrow B$$

SUCH THAT $\tilde{\varphi}(X) = b$ AND

$$\tilde{\varphi}\big|_A = \varphi.$$

PROOF. $\varphi : A \to B$ CAN BE EXTENDED TO A FUNCTION

$$\dot{\varphi} : A \cup \{X\} \longrightarrow B$$

BY $\dot{\varphi}(a) = \varphi(a) \quad \forall a \in A$

$\dot{\varphi}(X) = b$

$\dot{\varphi}$ EXTENDS TO AN ALG MAP

$$\Phi : F_{\mathbb{K}}(A \cup \{X\}) \longrightarrow B$$

$\Phi|_{A \cup \{X\}} = \dot{\varphi}$

$A\langle X\rangle$ "A ADJOIN X"

$$\Phi\left(a_1+a_2-(a_1+a_2)_A\right) =$$

$$= \Phi(a_1)+\Phi(a_2)-\Phi\Big(\underbrace{(a_1+a_2)_A}_{\in A}\Big) =$$

$$= \varphi(a_1)+\varphi(a_2)-\varphi(a_1+a_2) = 0$$

SINCE φ IS AN ALG MAP.

$$\Phi(1-1_A) = \Phi(1)-\Phi(\underbrace{1_A}_{\in A}) =$$

$$= 1_B - \varphi(1_A) = 0.$$

.... QED

RELATIONS.

DEF THE ALGEBRA OBTAINED FROM A BY ADJOINING X SUBJECT TO RELATIONS $R=0$ IS $\dfrac{A\langle X\rangle}{\langle R\rangle}$.

EX $A=\mathbb{k}[x]$

$$\mathbb{k}[x,y]=\frac{A\langle y\rangle}{\langle xy-yx\rangle}$$

EX. $A_1(\mathbb{k})=\mathbb{k}[x]\langle y\rangle/\langle yx-xy-1\rangle$

PROBLEM 1) WHAT IS THE RELATIONSHIP BETWEEN A-MODULES AND $A\langle X \rangle$-MODULES?

$$A\text{-}\underline{\mathrm{Mod}} \quad \text{vs.} \quad A\langle X \rangle\text{-}\underline{\mathrm{Mod}}\,?$$

$$M \rightsquigarrow \,?$$

$$? \leftsquigarrow N$$

PROBLEM 2) SHOW THAT ANY ALGEBRA IS ISOMORPHIC TO A QUOTIENT OF A FREE ALGEBRA.

PROBLEM 3) CONSTRUCT $U(\mathfrak{sl}_2(\Bbbk))$ FROM $\Bbbk[h]$ BY ADJOINING e, f IN TWO STEPS, SUBJECT TO RELATIONS.