PRESENTATIONS.

X ANY SET , $\mathbb{k}$ FIELD

$F(X) = F_{\mathbb{k}}(X)$ FREE ALGEBRA ON X

$$= \bigoplus_{\substack{w \text{ word} \\ \text{in } X}} \mathbb{k} w$$

$1 = \emptyset$ EMPTY WORD

$w = x_1 x_2 \cdots x_n$, $x_i \in X$, $n \geq 1$

MULTIPLICATON IN $F(X)$ IS BILINER EXTENSION OF JUXTAPOSITION (=CONCATENATION)

<u>EX</u> $X = \emptyset \Rightarrow F(X) = \mathbb{k}1 \cong \mathbb{k}$

$X = \{x\} \Rightarrow F(X) = \mathbb{k}[x]$

$$X = \{x, y\} \Longrightarrow F(X) = \underbrace{\mathbb{k}1}_{\deg 0} \oplus \underbrace{\mathbb{k}x \oplus \mathbb{k}y}_{\deg 1} \oplus$$

$$\oplus \underbrace{\mathbb{k}x^2 \oplus \mathbb{k}xy \oplus \mathbb{k}yx \oplus \mathbb{k}y^2}_{\deg 2} \oplus \cdots$$

DEF LET X BE A SET, AND R A SUBSET OF $F(X)$. THE ALGEBRA WITH GENERATORS X AND RELATIONS R IS

$$\Bbbk\langle X \mid R=0\rangle = \frac{F_{\Bbbk}(X)}{\langle R\rangle}$$

2-SIDED IDEAL GENERATED BY R.

THIS ALGEBRA IS SAID TO BE GIVEN BY A PRESENTATION.

Ex $X = \{x, y\}$

$R = \{xy - yx - 1\}$

THE WEYL ALGEBRA IS

$$A_1(\mathbb{k}) = \mathbb{k}\langle x, y \mid xy - yx - 1 \rangle =$$

$$= \frac{F_{\mathbb{k}}(\{x, y\})}{\langle xy - yx - 1 \rangle}$$

EX (RANDOM EX.)

$$X = \{x, y, z\}$$

$$R = \{ xy^2 - 3zy + yz - 4, \\ x + y - z^2 \}$$

WARNINGS: 1) POSSIBLE THAT $1 \in \langle R \rangle$!

WOULD LEAD TO THE ZERO ALGEBRA

$$1 = 0 \iff \dim A = 0$$

(ISOMORPHISM PROBLEM)

2) HARD TO KNOW WHEN $\Bbbk\langle X \mid R = 0\rangle \cong \Bbbk\langle Y \mid S = 0\rangle$

HOMOMORPHISMS FROM ALGEBRAS WITH GIVEN PRESENTATION.

UNIVERSAL PROPERTY OF FREE ALGEBRAS:

$$\left(\begin{array}{ccc} X & \rightsquigarrow & F(X) \\ \underline{\mathrm{Set}} & & \underline{\mathrm{Alg}} \end{array} \quad \text{FUNCTOR} \right)$$

TO CONSTRUCT A HOMOMORPHISM OF ALGEBRAS $\varphi : F(X) \longrightarrow A$, WHERE X IS A SET & A IS ANY ALGEBRA IT SUFFICES TO CHOOSE IMAGES $\varphi(x)$ FOR ALL $x \in X$. ANY CHOICE IS OK.

<u>Ex</u> $\varphi : F_{\mathbb{k}}(\{x,y\}) \longrightarrow U(\mathfrak{sl}_2(\mathbb{k}))$

THE ASSIGNMENT $\qquad e, f, h \in U(\mathfrak{sl}_2(\mathbb{k}))$

$$\begin{cases} \varphi(x) = ehh = eh^2 \\ \varphi(y) = feh + h \end{cases}$$

$$\Big(\varphi : \{x,y\} \to U(\mathfrak{sl}_2(\mathbb{k})) \Big)$$

EXTENDS UNIQUELY TO ALGEBRA HOMOMORPHISM.

$\mathcal{O} : \underline{\mathrm{Alg}} \longrightarrow \underline{\mathrm{Set}}$ Forgetful

$$\mathrm{Hom}_{\underline{\mathrm{Set}}}(X, \mathcal{O}(A)) \cong \mathrm{Hom}_{\underline{\mathrm{Alg}}}(F(X), A)$$

TO CONSTRUCT ALGEBRA MAP

$$\varphi : \mathbb{k}\langle X \mid R = 0 \rangle \longrightarrow A$$

1) PICK $\varphi(x)$ FOR ALL $x \in X$

2) GET UNIQUE EXTENSION TO ALGEBRA MAP $\tilde{\varphi} : F(X) \longrightarrow A$

3) CHECK $R \subset \operatorname{Ker} \tilde{\varphi}$

4) THEN $\langle R \rangle \subset \operatorname{Ker} \tilde{\varphi} \Longrightarrow$ GET INDUCED ALG MAP $\varphi : \dfrac{F(X)}{\langle R \rangle} \longrightarrow A$.

$\underline{EX}$. $A = \mathbb{k}\langle x \mid x^2 = 0 \rangle$ $\quad X = \{x\}$, $R = \{x^2\}$

WANT $\varphi : A \longrightarrow M_2(\mathbb{k})$

$\varphi' : \{x\} \longrightarrow M_2(\mathbb{k})$

$$\varphi'(x) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

GET $\tilde{\varphi} : F(\{x\}) = \mathbb{k}[x] \longrightarrow M_2(\mathbb{k})$

CHECK $x^2 \in \operatorname{Ker} \tilde{\varphi}$

$$\tilde{\varphi}(x^2) = (\tilde{\varphi}(x))^2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$A = \mathbb{k}1 \oplus \mathbb{k}\bar{x}$, $\bar{x} = x + \langle x^2 \rangle$

PROBLEMS

1. WEYL ALG $A_1(\mathbb{k}) = \mathbb{k}\langle x, y \mid yx - xy - 1 \rangle$

SHOW THAT THERE EXISTS A (UNIQUE) ALGEBRA MAP

$$\varphi : A_1(\mathbb{k}) \longrightarrow \mathrm{End}_{\mathbb{k}}(\mathbb{k}[x])$$

SATISFYING

$$\varphi(x) = L_x \quad , \quad L_x(P(x)) = x \cdot P(x)$$

$$\varphi(y) = D_x \qquad D_x(P(x)) = P'(x) = \frac{dP(x)}{dx}$$

$U(\mathfrak{sl}_2(\mathbb{k}))$ HAS A PRESENTATION

$$U(\mathfrak{sl}_2(\mathbb{k})) = \mathbb{k}\left\langle e, f, h \;\middle|\; \begin{array}{l} ef - fe - h = 0 \\ he - eh - 2e = 0 \\ hf - fh + 2f = 0 \end{array} \right\rangle$$

② SHOW THAT THERE IS AN ALG MAP

$$\varphi : U(\mathfrak{sl}_2(\mathbb{k})) \longrightarrow \mathrm{End}_{\mathbb{k}}(\mathbb{k}[x,y])$$

SATISFYING

$$\varphi(e) = x\frac{\partial}{\partial y} : P(x,y) \mapsto x \cdot \frac{\partial P}{\partial y}$$

$$\varphi(f) = y\frac{\partial}{\partial x} : P(x,y) \longmapsto y\frac{\partial P}{\partial x}$$

$$\varphi(h) = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}$$