JULY 22, 2020

$\mathbb{C}$ assoc. alg A

- vector space
- multiplication $\quad A \otimes A \longrightarrow A$

$$a \otimes b \longmapsto ab$$

$$(ab)c = a(bc)$$

$$\left\{ \begin{array}{c} \text{bilinear maps} \\ A \times A \to A \end{array} \right\} \xrightarrow{\simeq} \left\{ \begin{array}{c} \text{linear maps} \\ A \otimes A \to A \end{array} \right\}$$

- $1_A \in A \qquad 1_A a = a = a 1_A \quad \forall a \in A.$

($\mathbb{Z}$-algebra = ring.
k-algebras k comm. ring.)

Modules An left A-module V is a

- vector space with
- an action $A \otimes V \longrightarrow V$

$$a \otimes v \longmapsto a.v$$

$$(ab).v = a.(b.v)$$

- $1_A . v = v$

$\mathbb{C}$-module = vector space

$(A = \mathbb{C} = \mathbb{C} \cdot 1_A$ trivial alg$)$

Ex $V = A^n = \{(a_1, \dots, a_n) \mid a_i \in A\}$

$a \cdot (a_1, \dots, a_n) = (aa_1, \dots, aa_n)$

Ex G (finite) grp acting on a set X.

$A = \mathbb{C}G$ group algebra $= \{\sum_{g \in G} \lambda_g g \mid \lambda_g \in \mathbb{C}\}$

$V = \mathbb{C}X$ = vector sp. with basis X.

By bilinearity the action of G on X extends to an action of $\mathbb{C}G$ on $\mathbb{C}X$.

<u>$\mathbb{C}X$ becomes</u> a $\mathbb{C}G$-module.

A <u>representation</u> of an alg A is a pair (V, ρ) where

- V is a vector space, &
- $\rho : A \longrightarrow \mathrm{End}(V)$ alg map.

$$\left\{\begin{array}{c}\text{Reps } (V,\rho) \\ \text{of } A\end{array}\right\} \xrightarrow{\simeq} \left\{\begin{array}{c}A\text{-modules} \\ V\end{array}\right\}$$

$$(V,\rho) \longmapsto V$$

$$a \cdot v = \rho(a) v$$

$$\left(\begin{array}{c} G \cong G^{op} \\ g \mapsto g^{-1} \end{array}\right)$$

$$\rho : A \to \mathrm{End}(V)^{op}$$

$$a \cdot_{op} b = ba$$

Module term	Representation term
Module	Representation
Simple module	Irreducible Rep
Indecomposable module $V \not\cong U_1 \oplus U_2$	Indec. Rep.
Semisimple modules $V = \bigoplus_i V_i$ (V_i simples)	Completely reducible reps.
A-module map $\varphi : V \to W$, $\varphi(a.v) = a.\varphi(v)$	Intertwining operators. $\varphi : V \to W$

Examples of assoc. algebras.

- $\mathrm{End}(V)$; $M_n(\mathbb{C})$
- $\mathbb{C}G$, $|G| < \infty$
- $U(\mathfrak{g})$, $\mathfrak{g}$ Lie algebras
- (T) GWAs
- quantum groups
- Affine Hecke algs.

⋮

Finite-dimensional modules are much better understood.

Lie algebras.

Lie group $\rightsquigarrow$ Lie algebra

= group & a manifold

$\bigcirc \; U(1) \longmapsto \mathbb{R}, \; [x,y]=0 \;\; \forall x,y$

$= \{z \mid |z|=1\}$

<u>DEF</u> A **Lie alg** $\mathfrak{g}$ is:

- a vector space with
- a linear map $\mathfrak{g} \otimes \mathfrak{g} \longrightarrow \mathfrak{g}$ called the <u>bracket</u> $x \otimes y \longmapsto [x,y]$

such that

- $[x,x]=0 \xRightarrow{\text{char} \neq 2} [x,y]=-[y,x]$
- $[x,[y,z]]+[y,[z,x]]+[z[x,y]]=0$ (Jacobi Id.) $\forall x,y,z$

<u>EX</u>. A assoc. alg., $[x,y]=xy-yx$ makes A a Lie alg.

<u>Recall:</u> G group $\rightsquigarrow$ $\mathbb{C}G$ assoc. alg.

G-Modules (= Reps of G) $\longleftrightarrow$ $\mathbb{C}G$-module

$\mathfrak{g}$ Lie alg $\rightsquigarrow$ $\mathcal{U}(\mathfrak{g})$ assoc alg.
(universal) enveloping alg of $\mathfrak{g}$.

$\mathfrak{g}$-Modules (= Reps of $\mathfrak{g}$) $\longleftrightarrow$ $\mathcal{U}(\mathfrak{g})$-modules.

Ex $U(\mathfrak{gl}_n)$

$$A = M_n(\mathbb{C}) = \{n\times n\text{-matrices}\}$$

an assoc. alg.

$\rightsquigarrow$ $\mathfrak{gl}_n = M_n(\mathbb{C})$ with $[x,y] = xy - yx$ for all $n\times n$-matrices x,y

general linear Lie alg

$\left(\mathfrak{sl}_n = \{ x \in \mathfrak{gl}_n \mid \mathrm{Tr}(x) = 0 \} \right)$

special linear Lie alg.

$$\rightsquigarrow\ U(\mathfrak{gl}_n) = \frac{T(\mathfrak{gl}_n)}{\langle x\otimes y - y\otimes x - [x,y] \mid x,y\in\mathfrak{gl}_n \rangle}$$

$U(\mathfrak{gl}_n) \ni \sum \lambda_i x_{i_1} \cdots x_{i_j}$, $x_a \in \mathfrak{gl}_n$

Open problems

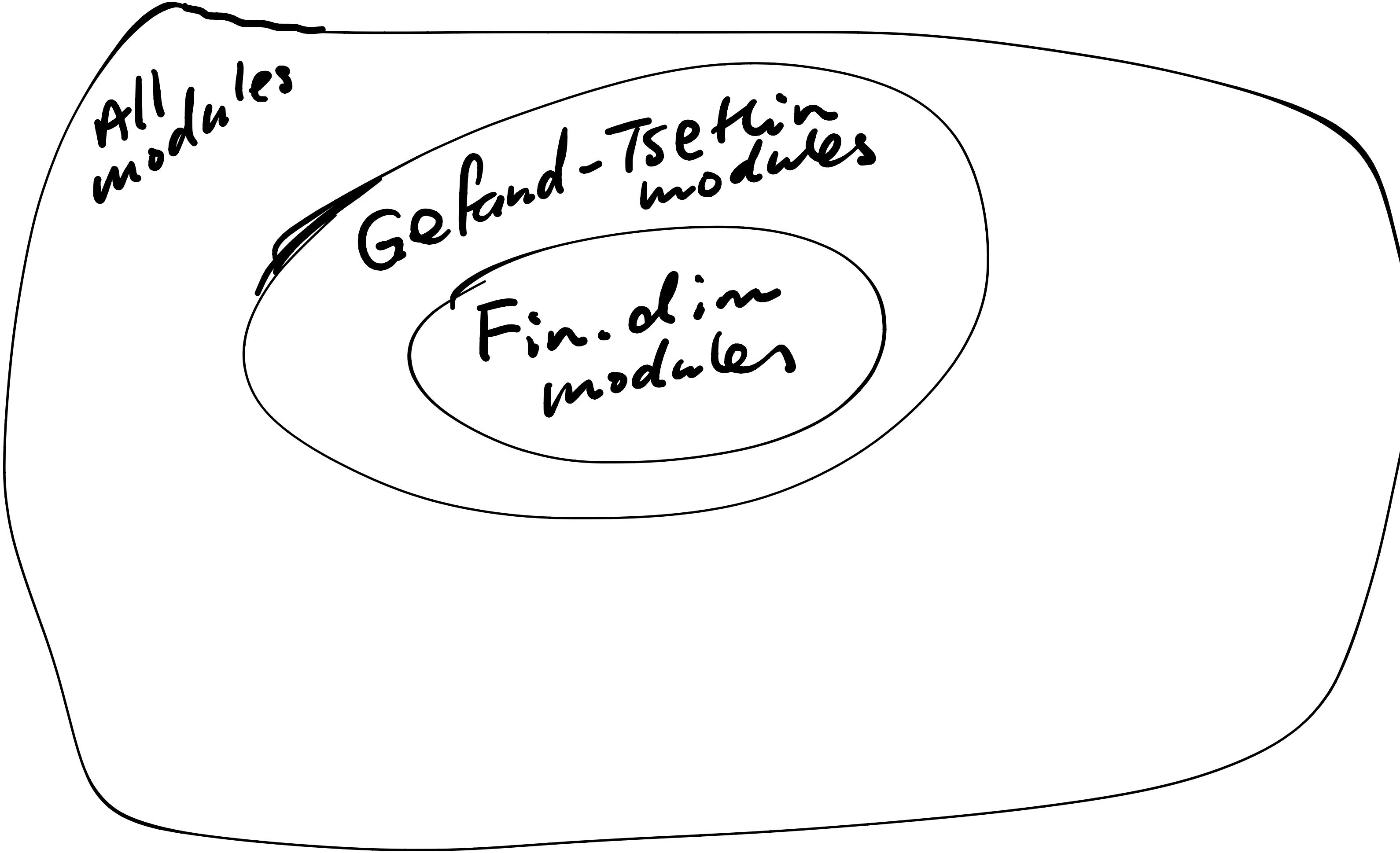
crazy

1) Classify all simple $U(\mathfrak{gl}_n)$-modules.

2) Given two simple Gelfand–Tsetlin modules over $U(\mathfrak{gl}_n)$, V, W find the composition factors of $V \otimes W$.

$$x.(v \otimes w) = (x.v) \otimes w + v \otimes (x.w)$$

$x \in \mathfrak{gl}_n$


All modules
Gelfand-Tsetlin modules
Fin. dim modules