MATH 317, SECTION B — NAME: ____________

FINAL EXAM, 12 DEC 2017 — SEC.: ________

The exam concludes after 2 hours. You must show all of your work.
You may use any calculator without wireless capability and may not use memory.
You may use the back of the page.

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1. (15 points) Verify that the Cauchy-Schwarz inequality holds with respect to the $\mathbb{R}^3$ vectors $[3, 0, -4]$ and $[-2, 2, 1]$.

Solution: Cauchy-Schwarz says $|\mathbf{v} \cdot \mathbf{w}| \leq \|\mathbf{v}\| \|\mathbf{w}\|$.

$$|[3, 0, -4] \cdot [-2, 2, 1]| = |(3)(-2) + (0)(2) + (-4)(1)| = |-10| = 10$$
$$\|[3, 0, -4]\| \, \|[-2, 2, 1]\| = \sqrt{3^2 + 0^2 + (-4)^2}\sqrt{(-2)^2 + 2^2 + 1} = (5)(3) = 15$$

Since $10 \leq 15$, the Cauchy-Schwarz inequality is satified.

2. (15 points) Give the inverse of the matrix $\mathbf{A} = \begin{bmatrix} -6 & 2 \\ -3 & 4 \end{bmatrix}$.

Solution: If $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$\begin{aligned} \mathbf{A} &= \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \\ &= \frac{1}{(-6)(4) - (2)(-3)} \begin{bmatrix} 4 & -2 \\ 3 & -6 \end{bmatrix} \\ &= \frac{1}{-18} \begin{bmatrix} 4 & -2 \\ 3 & -6 \end{bmatrix} \\ &= \begin{bmatrix} -2/9 & 1/9 \\ -1/6 & 1/3 \end{bmatrix} \end{aligned}$$

This can also be accomplished by row reduction.

3. Let $\mathbf{D} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}$, where

$$\mathbf{D} = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \quad \text{and} \quad \mathbf{P} = \begin{bmatrix} 1 & 1 & 3 & 2 \\ -2 & 0 & 7 & 8 \\ 1 & 0 & -3 & -4 \\ 1 & 1 & 2 & 3 \end{bmatrix}$$

(a) (6 points) What are the eigenvalues of $\mathbf{A}$?

Solution: Because $\mathbf{A}$ is diagonalizable with diagonal matrix $\mathbf{D}$, the eigenvalues of $\mathbf{A}$ are the same as those of $\mathbf{D}$. These eigenvalues are $\{2, -1, 3\}$.

(b) (8 points) Give a basis for each of the eigenspaces for each eigenvalue in part (a).

Solution:

$$\begin{aligned} E_2 &= \text{span}(\{[1, -2, 1, 1]\}) \\ E_{-1} &= \text{span}(\{[1, 0, 0, 1], [3, 7, -3, 2]\}) \\ E_3 &= \text{span}(\{[2, 8, -4, 3]\}) \end{aligned}$$

(c) (6 points) What is $|\mathbf{A}|$?

Solution:

$$|\mathbf{A}| = |\mathbf{P}||\mathbf{D}||\mathbf{P}^{-1}| = |\mathbf{P}||\mathbf{D}|\frac{1}{|\mathbf{P}|} = |\mathbf{D}| = (2)(-1)(-1)(3) = 6$$

4. For the subset $S = \{[3, 5, -3], [-2, -4, 3], [1, 2, -1]\}$ of $\mathbb{R}^3$.

   (a) (10 points) Use the Simplified Span Method to find a simplifed form for the vectors in span(S). Does S span $\mathbb{R}^3$?

Solution: The Simplified Span Method has us put the vectors as rows in a matrix and row reduce:

$$\begin{bmatrix} 3 & 5 & -3 \\ -2 & -4 & 3 \\ 1 & 2 & -1 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The simplified basis is $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. Yes, S does span $\mathbb{R}^3$.

   (b) (10 points) Give a basis for span(S). What is dim(span(S))?

Solution: As in part (a), $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is a basis for span(S) (so is S itself). The dimension of span(S) is 3.

5. (15 points) Let $\mathbf{B}$ and $\mathbf{C}$ be fixed $n \times n$ matrices, with $\mathbf{B}$ nonsingular. Prove that the mapping $f : \mathcal{M}_{nn} \to \mathcal{M}_{nn}$ given by $f(\mathbf{A}) = \mathbf{CAB}^{-1}$ is a linear operator.

Solution: Since f maps from one vector space to itself, it remains to show that f is a linear transformation. Let $\mathbf{A}_1, \mathbf{A}_2 \in \mathcal{M}_{nn}$ and $c \in \mathbb{R}$.

$$\begin{aligned} f(\mathbf{A}_1 + \mathbf{A}_2) &= \mathbf{C}(\mathbf{A}_1 + \mathbf{A}_2)\mathbf{B}^{-1} = \mathbf{CA}_1\mathbf{B}^{-1} + \mathbf{CA}_2\mathbf{B}^{-1} && = f(\mathbf{A}_1) + f(\mathbf{A}_2) \\ f(c\mathbf{A}_1) &= \mathbf{C}(c\mathbf{A})\mathbf{B}^{-1} = c\mathbf{CA}_1\mathbf{B} && = cf(\mathbf{A}_1) \end{aligned}$$

Thus, f is a linear transformation and, hence, a linear operator.

6. (15 points) Consider $L : \mathcal{P}_3 \to \mathcal{M}_{22}$ given by $L(ax^3 + bx^2 + cx + d) = \begin{bmatrix} a-d & 2b \\ b & c+d \end{bmatrix}$. What is $\dim(\ker(L))$? What is $\dim(\text{range}(L))$?

Solution: In order for $L(\mathbf{p}(x)) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, the following must be satisfied:

$$\begin{cases} a - d = 0 \\ 2b = 0 \\ b = 0 \\ c + d = 0 \end{cases}$$

The solution is $d = a$, $c = -a$, and $b = 0$ and so the $\ker(L) = \text{span}(\{x^3 - x + 1\})$. Hence $\dim(\ker(L)) = 1$ and $\dim(\text{range}(L)) = 4 - 1 = 3$.

7. Either find a matrix A with the following properties or explain why no such matrix exists:

(a) (10 points) The kernel contains $\begin{bmatrix}1\\0\\1\end{bmatrix}$, $\begin{bmatrix}1\\2\\2\end{bmatrix}$ and the rowspace contains $\begin{bmatrix}1 & 1 & -1\end{bmatrix}$.

> **Solution:** Since the row space is orthogonal to the null space, there can be no such matrix since the vectors $\begin{bmatrix}1\\2\\2\end{bmatrix}$, $\begin{bmatrix}1 & 1 & -1\end{bmatrix}$ are not orthogonal. That is, any vector in the kernel is orthogonal to every row. But that forces every vector in the kernel to be orthogonal to every vector in the rowspace.

(b) (10 points) The column space and kernel both have basis $\begin{bmatrix}1\\0\\1\end{bmatrix}$, $\begin{bmatrix}1\\2\\3\end{bmatrix}$.

> **Solution:** The matrix is 3×3. By the Dimension Theorem, $3 = \dim(\mathbf{A}) = \dim(\ker(\mathbf{A})) + \dim(\text{range}(\mathbf{A}))$. Both the kernel and range having dimension 2 violates the Dimension Theorem.

8. (20 points) Let $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m$ be a set of linearly independent vectors in $\mathbb{R}^n$, where $m > 1$. For $k = 1, 2, \ldots, m-1$, let $\mathbf{w}_k = \mathbf{x}_k + \mathbf{x}_{k+1}$. Prove that the vectors $\mathbf{w}_1, \mathbf{w}_2, \ldots \mathbf{w}_{m-1}$ are linearly independent. Your proof should use the definition of linear independence.

> **Solution:** If $c_1\mathbf{w}_1 + c_2\mathbf{w}_2 + \cdots + c_{m-1}\mathbf{w}_{m-1} = 0$ then we get $c_1\mathbf{x}_1 + (c_1 + c_2)\mathbf{x}_2 + \cdots + (c_{m-2} + c_{m-1})\mathbf{x}_{m-1} = 0$. Now using linear independence of the $\mathbf{x}_k$'s, we get $c_1 = 0$, $(c_1 + c_2) = 0$,$\ldots(c_{m-2} + c_{m-1}) = 0$. It follows that $c_i = 0$ for $i = 1, 2, \ldots, m-1$. Thus the vectors $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_{m-1}$ are linearly independent.

9. Consider the following orthogonal set $S = \{[1, 3, -2], [5, 1, 4]\}$.

(a) (15 points) Enlarge S to an orthogonal basis for $\mathbb{R}^3$.

Solution: Append the standard basis and use the Enlarging Method.

$$\begin{bmatrix} 1 & 5 & 1 & 0 & 0 \\ 3 & 1 & 0 & 1 & 0 \\ -2 & 4 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 0 & 2/7 & -1/14 \\ 0 & 1 & 0 & 1/7 & 3/14 \\ 0 & 0 & 1 & -1 & -1 \end{bmatrix}$$

So, a basis is $([1, 3, -2], [5, 1, 4], [1, 0, 0])$. We use Gram-Schmidt on this set.

$$\begin{aligned} \mathbf{v}_1 &= [1, 3, -2] \\ \mathbf{v}_2 &= [5, 1, 4] \\ \mathbf{v}_3 &= [1, 0, 0] - \frac{[1, 0, 0] \cdot [1, 3, -2]}{[1, 3, -2] \cdot [1, 3, -2]}[1, 3, -2] - \frac{[1, 0, 0] \cdot [5, 1, 4]}{[5, 1, 4] \cdot [5, 1, 4]}[5, 1, 4] \\ &= [1, 0, 0] - \frac{1}{14}[1, 3, -2] - \frac{5}{42}[5, 1, 4] \\ &= \frac{1}{3}[1, -1, -1] \end{aligned}$$

(b) (15 points) Let $\mathcal{W} = \text{span}(S)$. Find the minimum distance between the point $P = (1, 1, 1)$ and the subspace $\mathcal{W}$.

Solution: Let $\mathbf{v} = [1, 1, 1]$. The orthonormal basis is $\left(\frac{1}{\sqrt{14}}[1, 3, -2], \frac{1}{\sqrt{42}}[5, 1, 4]\right)$. Hence,

$$\begin{aligned} \text{proj}_{\mathcal{W}}\mathbf{v} &= \left([1, 1, 1] \cdot \frac{1}{\sqrt{14}}[1, 3, -2]\right)\left(\frac{1}{\sqrt{14}}[1, 3, -2]\right) + \left([1, 1, 1] \cdot \frac{1}{\sqrt{42}}[5, 1, 4]\right)\left(\frac{1}{\sqrt{42}}[5, 1, 4]\right) \\ &= \frac{2}{3}[2, 1, 1] \end{aligned}$$

Thus,

$$\|\mathbf{v} - \text{proj}_{\mathcal{W}}\mathbf{v}\| = \left\|[1, 1, 1] - \frac{2}{3}[2, 1, 1]\right\| = \frac{\sqrt{3}}{3}$$

10. **TRUE** or **FALSE**. No explanation needed. You must write out each word. If we can't decipher your answer, we will mark it wrong.

(a) (3 points) ______

If $\mathbf{A}$ is an $m \times n$ matrix and $\mathbf{B}$ is an $n \times 1$ matrix, then $\mathbf{AB}$ is an $m \times 1$ matrix representing a linear combination of the columns of $\mathbf{A}$.

Solution: TRUE.

If the columns of $\mathbf{A}$ are $\mathbf{a}_1, \ldots, \mathbf{a}_n$ and $\mathbf{B} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$, then $\mathbf{AB}$ is $b_1\mathbf{a}_1 + \cdots + b_n\mathbf{a}_n$.

(b) (3 points) ______

If $\mathbf{A}$ and $\mathbf{B}$ are $n \times n$ matrices such that $\mathbf{AB}$ is nonsingular, then $\mathbf{A}$ is nonsingular.

Solution: TRUE.
Computing the determinant, $0 \neq |\mathbf{AB}| = |\mathbf{A}||\mathbf{B}|$.

(c) (3 points) ______

Any two $n \times n$ matrices having the same nonzero determinant are row equivalent.

Solution: TRUE.
Any $n \times n$ matrix with a nonzero determinant is row equivalent to $\mathbf{I}_n$, thus they are row equivalent to each other.

(d) (3 points) ______

If B and C are ordered bases for a finite dimensional vector space $\mathcal{V}$ and if $\mathbf{P}$ is the transition matrix from B to C, then $\mathbf{P}^T$ is the transition matrix from C to B.

Solution: FALSE.
The transition matrix from C to B should be $\mathbf{P}^{-1}$.

(e) (3 points) ______

There is a linear operator on $\mathbb{R}^5$ such that $\ker(L) = \text{range}(L)$.

> **Solution:** FALSE.
> If they are equal, they have the same dimension, which should sum to 5. However, that would require each of them to have dimension $5/2$, which is impossible.

(f) (3 points) ______

If $L : \mathcal{V} \to \mathcal{W}$ is an isomorphism and $M : \mathcal{W} \to \mathcal{X}$ is a linear transformation, then $\text{range}(M \circ L) = \text{range}(M)$.

> **Solution:** TRUE.
> It is clear that $\text{range}(M) \supseteq \text{range}(M \circ L)$. Since L is an isomorphism, for every $\mathbf{w} \in \mathcal{W}$, there is a $\mathbf{v} \in \mathcal{V}$ such that $L(\mathbf{v}) = \mathbf{w}$. Thus, for every $\mathbf{x} \in \text{range}(L)$, there is a $\mathbf{w} \in \mathcal{W}$ with $M(\mathbf{w}) = \mathbf{x}$ and hence a $\mathbf{v} \in \mathcal{V}$ such that $M(L(\mathbf{v})) = \mathbf{x}$.

(g) (3 points) ______

There is a vector space $\mathcal{V}$ such that $\{\mathbf{0}_{\mathcal{V}}\}$ is a basis for $\mathcal{V}$.

> **Solution:** FALSE.
> By definition, $\mathbf{0}_{\mathcal{V}}$ can be in no linearly independent set. If $\mathcal{V} = \{\mathbf{0}_{\mathcal{V}}\}$, then the basis of $\mathcal{V}$ is the empty set $\{\}$.

(h) (3 points) ______

If L is a linear operator on a nontrivial finite dimensional vector space $\mathcal{V}$, x^2 is a factor of $p_L(x)$, and $\dim(E_0) = 1$, then L is not diagonalizable.

> **Solution:** TRUE.
> The algebraic multiplicity of eigenvalue 0 is at least two because x^2 divides $p_L(x)$ but the geometric multiplicity is 1. So, L is not diagonalizable.

(i) (3 points) ____________
Every orthogonal matrix is nonsingular.

Solution: TRUE.
The inverse of an orthogonal $\mathbf{A}$ is, by definition, $\mathbf{A}^T$.

(j) (3 points) ____________
If $\mathcal{W}$ is a subspace of $\mathbb{R}^n$, and $L : \mathbb{R}^n \to \mathbb{R}^n$ is given by $L(\mathbf{v}) = \mathbf{proj}_{\mathcal{W}^\perp}\mathbf{v}$, then $\ker(L) = \mathcal{W}$.

Solution: TRUE.
By definition, $\ker(L) \supseteq \mathcal{W}$. Every $\mathbf{v} \in \mathcal{W}$ is in range(L) because $L(\mathbf{v}) = \mathbf{v}$ for all $\mathbf{v} \in \mathcal{W}$. So, $\mathcal{W} = \text{range}(L)$.

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